



Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Processes

Part B: Context-Free Languages

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<https://ths.rwth-aachen.de/teaching/ws18/b-it-bridging-course/>

Context-Free Grammars and Languages

Outline of Part B

Context-Free Grammars and Languages

Context-Free vs. Regular Languages

The Word Problem for CFLs

The Emptiness Problem for CFLs

Closure Properties of CFLs

Pushdown Automata

Outlook

Context-Free Grammars and Languages

Introductory Example I

Example B.1

Syntax definition of programming languages by “Backus-Naur” rules

Here: **simple arithmetic expressions**

$$\begin{array}{l} \langle \textit{Expression} \rangle ::= 0 \\ \quad \quad \quad | 1 \\ \quad \quad \quad | \langle \textit{Expression} \rangle + \langle \textit{Expression} \rangle \\ \quad \quad \quad | \langle \textit{Expression} \rangle * \langle \textit{Expression} \rangle \\ \quad \quad \quad | (\langle \textit{Expression} \rangle) \end{array}$$

Meaning:

An expression is either 0 or 1, or it is of the form $u + v$, $u * v$, or (u) where u, v are again expressions

Introductory Example II

Example B.2 (continued)

Here we abbreviate $\langle \textit{Expression} \rangle$ as E , and use “ $\rightarrow$ ” instead of “ $::=$ ”. Thus:

$$E \rightarrow 0 \mid 1 \mid E + E \mid E * E \mid (E)$$

Introductory Example II

Example B.2 (continued)

Here we abbreviate $\langle \textit{Expression} \rangle$ as E , and use “ $\rightarrow$ ” instead of “ $::=$ ”. Thus:

$$E \rightarrow 0 \mid 1 \mid E + E \mid E * E \mid (E)$$

Now expressions can be generated by replacing nonterminal symbols according to rules, beginning with the start symbol E :

$$E \Rightarrow E * E$$

Introductory Example II

Example B.2 (continued)

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Now expressions can be generated by replacing nonterminal symbols according to rules, beginning with the start symbol E :

$$\begin{aligned} E &\Rightarrow E * E \\ &\Rightarrow (E) * E \end{aligned}$$

Context-Free Grammars and Languages

Introductory Example II

Example B.2 (continued)

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$$\begin{aligned} E &\Rightarrow E * E \\ &\Rightarrow (E) * E \\ &\Rightarrow (E) * 1 \end{aligned}$$

Introductory Example II

Example B.2 (continued)

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Context-Free Grammars and Languages

Introductory Example II

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Context-Free Grammars and Languages

Introductory Example II

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Now expressions can be generated by replacing nonterminal symbols according to rules, beginning with the start symbol E :

$$\begin{aligned} E &\Rightarrow E * E \\ &\Rightarrow (E) * E \\ &\Rightarrow (E) * 1 \\ &\Rightarrow (E + E) * 1 \\ &\Rightarrow (0 + E) * 1 \\ &\Rightarrow (0 + 1) * 1 \end{aligned}$$

$stm ::= stm; stm \mid stm \parallel stm \mid \text{if } cond \text{ then } stm \text{ else } stm$
 $\quad \text{while } cond \text{ do } stm \text{ od} \mid$
 $\quad \text{type } var \mid var := expr$

$cond ::= expr = expr \mid expr < expr \dots$

$expr ::= var \mid const \mid expr + expr \mid expr * expr \dots$

$var ::= [a-z]^+ ([a-z][0-9])^*$

$const ::= [0-9]^+ [0-9]^*$

Context-Free Grammars and Languages

Context-Free Grammars I

Definition B.3

A **context-free grammar (CFG)** is a quadruple

$$G = \langle N, \Sigma, P, S \rangle$$

where

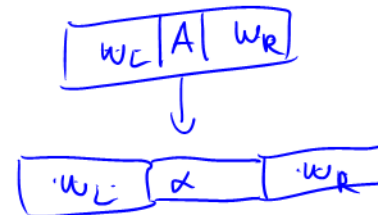
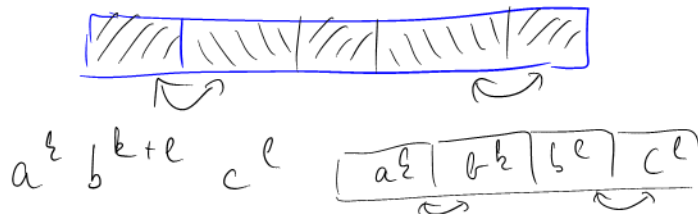
N is a finite set of **nonterminal symbols**

Σ is the (finite) alphabet of **terminal symbols** (disjoint from N)

P is a finite set of **production rules** of the form $A \rightarrow \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$

$S \in N$ is a **start symbol**

only one symbol!



Context-Free Grammars and Languages

Context-Free Grammars II

Example B.4

For the above example, we have:

$$N = \{E\}$$

$$\Sigma = \{0, 1, +, *, (,)\}$$

$$P = \{E \rightarrow \underline{0}, E \rightarrow \underline{1}, E \rightarrow \underline{E + E}, E \rightarrow \underline{E * E}, E \rightarrow \underline{(E)}\}$$

$$S = \underline{E}$$

$E \in \Sigma$

Terminal symbols are not further re-written

$0 \rightarrow \dots$

$+$ $\rightarrow \dots$

$E \rightarrow \dots$ ✓

$E \in N$

Termination? $\Sigma \neq \emptyset$

$$S \rightarrow A$$

$$A \rightarrow S$$

$$T \rightarrow a$$

S
|
A
|
S
|
A
|
S
|
⋮

generated language?

$$\boxed{S} \rightarrow A \quad S \rightarrow B$$

$$A \rightarrow B$$



Context-Free Grammars and Languages

Context-Free Languages I

Definition B.5

Let $G = \langle N, \Sigma, P, S \rangle$ be a CFG.

A **sentence** $\gamma \in (N \cup \Sigma)^*$ is **directly derivable** from $\beta \in (N \cup \Sigma)^*$ if there exist $\pi = \boxed{A \rightarrow \alpha} \in P$ and $\delta_1, \delta_2 \in (N \cup \Sigma)^*$ such that $\beta = \delta_1 \underline{A} \delta_2$ and $\underline{\gamma} = \delta_1 \underline{\alpha} \delta_2$ (notation: $\beta \xRightarrow{\pi} \gamma$ or just $\beta \Rightarrow \gamma$).

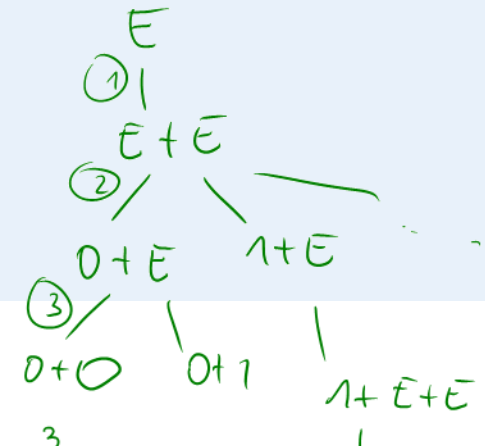
A **derivation** (of length n) of γ from β is a sequence of direct derivations of the form $\delta_0 \Rightarrow \delta_1 \Rightarrow \dots \Rightarrow \delta_n$ where $\delta_0 = \beta$, $\delta_n = \gamma$, and $\delta_{i-1} \Rightarrow \delta_i$ for every $1 \leq i \leq n$ (notation: $\beta \Rightarrow^* \gamma$).

A word $w \in \Sigma^*$ is called **derivable** in G if $S \Rightarrow^* w$.

$$\beta \quad (E) \quad \Rightarrow^* \quad (0+0) \quad \sigma$$

$$E \xRightarrow{E \rightarrow E+E} E+E \xRightarrow{E \rightarrow 0} 0+E \xRightarrow{E \rightarrow 0} 0+0$$

$$\beta = \delta_0 \Rightarrow \delta_1 \Rightarrow \delta_2 \Rightarrow \delta_3 = \sigma$$



$$E \Rightarrow^3 0+0$$

$$E \Rightarrow^5 1+0+1$$

$$1+0+E$$

$$1+0+1$$

Rules :

$S \rightarrow A$

$A \rightarrow a$

$A \rightarrow A$

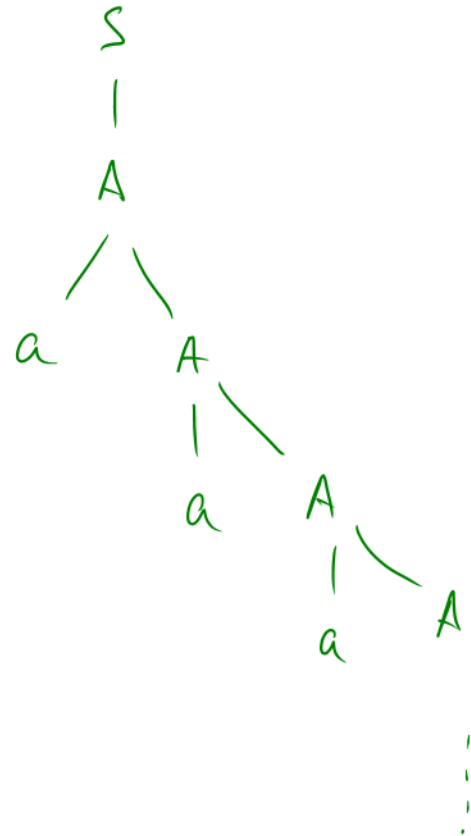
Derivations :

$S \Rightarrow^2 a$

$S \Rightarrow^3 a$

$S \Rightarrow^4 a$

$\vdots$



$$S \rightarrow AB$$

$$A \rightarrow a$$

$$B \rightarrow b$$

$$S \Rightarrow \boxed{AB} \Rightarrow aB \Rightarrow ab$$

$$\Rightarrow Ab \Rightarrow ab$$

$$\boxed{\quad \quad \quad \mid \text{LHS} \mid \quad \quad \quad} \in (\Sigma \cup \mathcal{N})^*$$

$$\uparrow$$

$$\boxed{\text{RHS}}$$

$$\hookrightarrow \boxed{\quad \quad \quad \mid \text{RHS} \mid \quad \quad \quad}$$

$$A \Rightarrow \overset{(*)}{\curvearrowright} B$$

in ≥ 0 steps

$$A \Rightarrow \overset{(+)}{\curvearrowright} B$$

in > 0 steps

$$A \Rightarrow \overset{(b)}{\curvearrowright} B$$

in k steps

Context-Free Grammars and Languages

Context-Free Languages I

Definition B.5

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A word $w \in \Sigma^*$ is called **derivable** in G if $S \Rightarrow^* w$.

The **language generated by G** is $L(G) := \{w \in \Sigma^* \mid S \Rightarrow^* w\}$.

A language $L \subseteq \Sigma^*$ is called **context-free (CFL)** if it is generated by some CFG.

Two grammars G_1, G_2 are **equivalent** if $L(G_1) = L(G_2)$.

$$\Sigma = \{0, 1\}$$

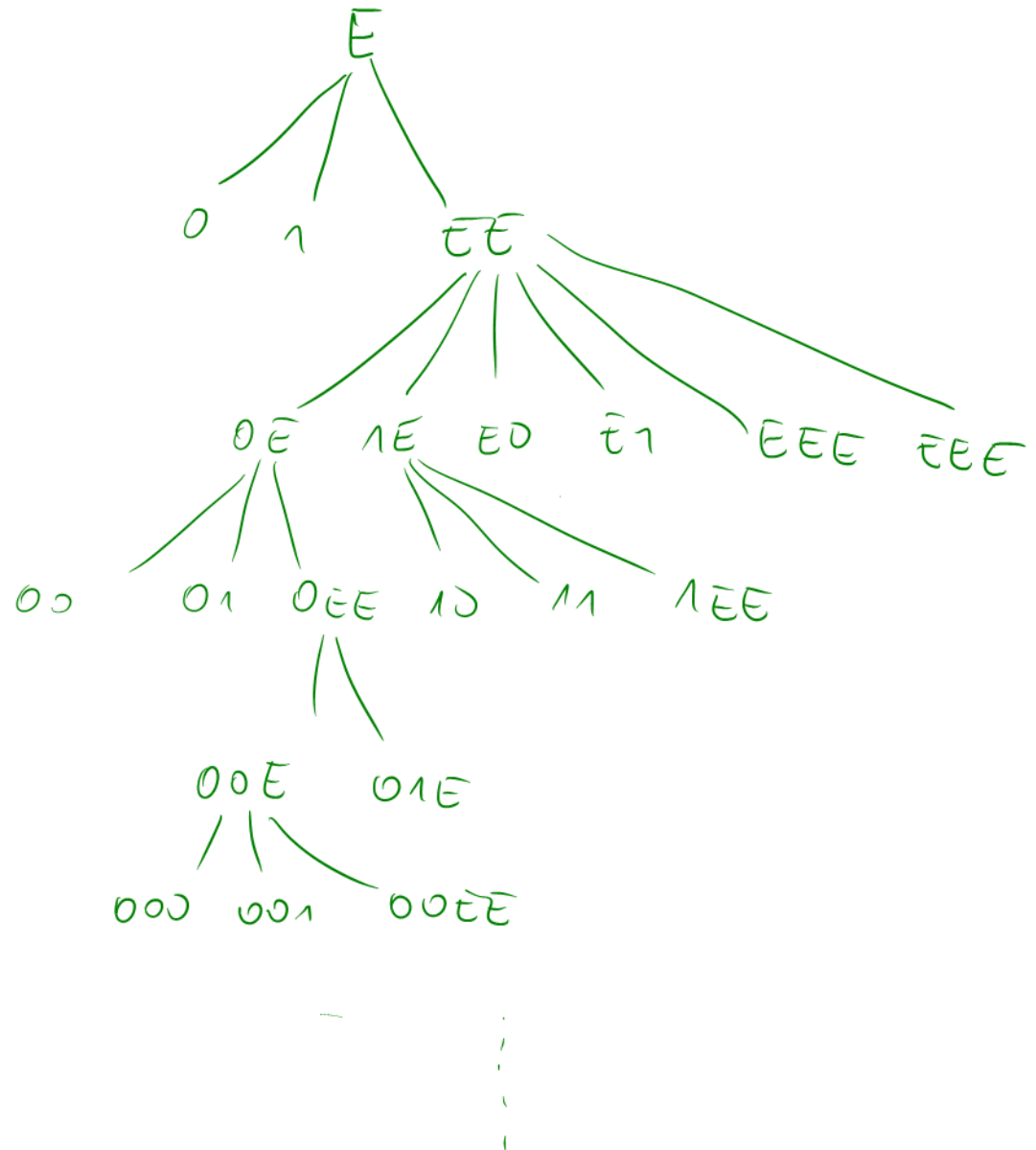
$$E \rightarrow 0$$

$$E \rightarrow 1$$

$$E \rightarrow EE$$

$$\rightarrow 0 \xrightarrow{0,1} (0) 0,1$$

$$(0 + 1)^+$$



Context-Free Languages II

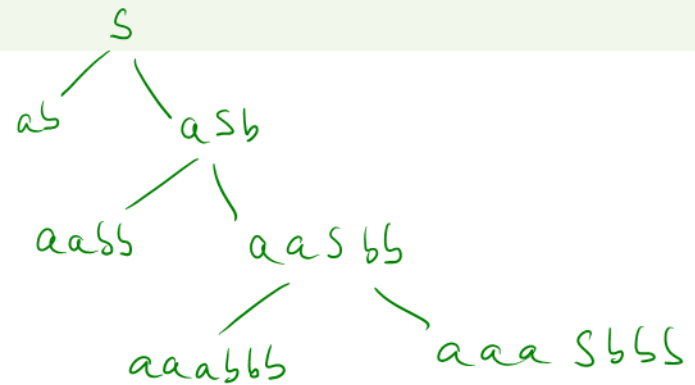
The language $\{a^n b^n \mid n \geq 1\}$ is context-free. It is generated by the grammar $G = \langle N, \Sigma, P, S \rangle$ with

$$\Sigma = \{a, b\}$$

(proof: generating $a^n b^n$ requires exactly $n - 1$ applications of the first and one concluding application of the second rule)

$$S \rightarrow a S b$$

$$S \rightarrow ab$$



Problem: $a^n b^n c^n$

$$\boxed{a^n}$$

$$A \rightarrow aA \mid a$$

$$a^* = \{a^n \mid n \in \mathbb{N}_0\}$$

$$a^* b^*$$

$$\bigcup_{i=0}^{\infty}$$

$$a^i b^i$$

$$\boxed{a^n b^n} \neq a^* b^*$$

$$a^n \equiv a^*$$

Context-Free Languages II

Example B.6

The language $\{a^n b^n \mid n \geq 1\}$ is context-free. It is generated by the grammar $G = \langle N, \Sigma, P, S \rangle$ with

$$N = \{S\}$$

$$\Sigma = \{a, b\}$$

$$P = \{S \rightarrow aSb \mid ab\}$$

(proof: generating $a^n b^n$ requires exactly $n - 1$ applications of the first and one concluding application of the second rule)

Remark: illustration of derivations by **derivation trees**

- root labelled by start symbol

- leaves labelled by terminal symbols

- successors of node labelled according to right-hand side of production rule

(example on the board)

Context-Free Grammars and Languages

Seen:

Context-free grammars

Derivations

Context-free languages

Context-Free Grammars and Languages

Seen:

- Context-free grammars

- Derivations

- Context-free languages

Open:

- Relation between context-free and regular languages

Context-Free vs. Regular Languages

Outline of Part B

Context-Free Grammars and Languages

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The Emptiness Problem for CFLs

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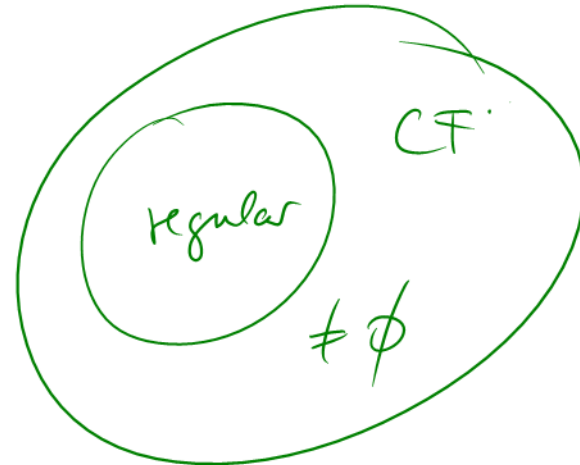
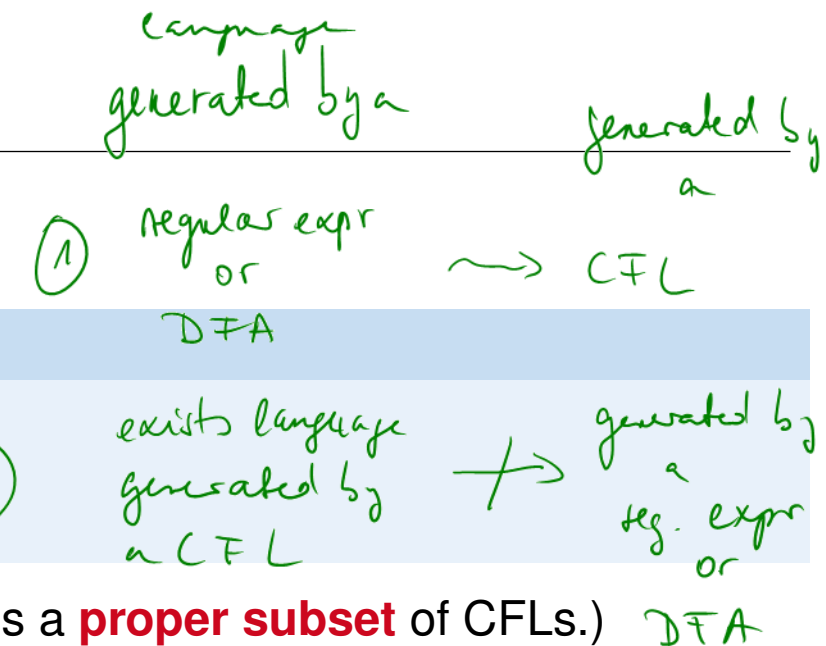
Context-Free vs. Regular Languages

Context-Free vs. Regular Languages

Theorem B.7

1. Every regular language is context-free.
2. There exist CFLs which are not regular.

(In other words: the class of regular languages is a **proper subset** of CFLs.)



Context-Free vs. Regular Languages

Context-Free vs. Regular Languages

Theorem B.7

1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(In other words: the class of regular languages is a **proper subset** of CFLs.)

Proof.

1. Let L be a regular language, and let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA which recognises L . $G := \langle N, \Sigma, P, S \rangle$ is defined as follows:
 - $N := Q, S := q_0$
 - if $\delta(q, a) = q'$, then $q \rightarrow aq' \in P$
 - if $q \in F$, then $q \rightarrow \varepsilon \in P$

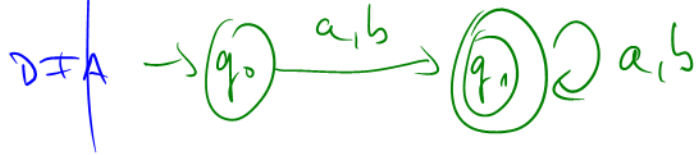
Obviously a w -labelled run in $\mathfrak{A}$ from q_0 to F corresponds to a derivation of w in G , and vice versa. Thus $L(\mathfrak{A}) = L(G)$ (example on the board).

2. An example is $\{a^n b^n \mid n \geq 1\}$ (see Ex. B.6).



regular language

reg. expr. a^*b^* $\Sigma = \{a, b\}$

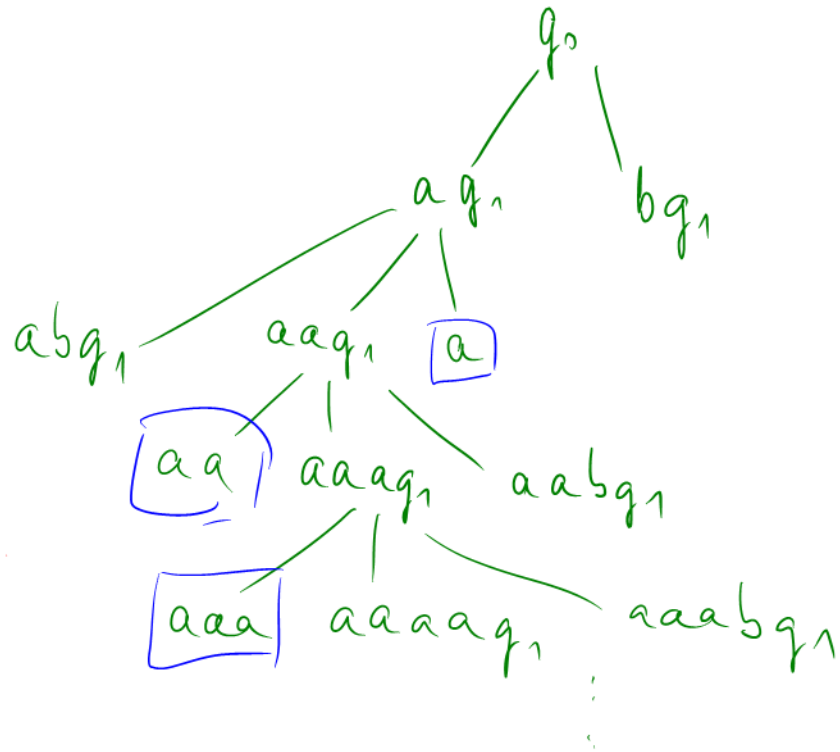


$$N = \{q_0, q_1\}$$

$$\Sigma = \{a, b\}$$

$$S = q_0$$

$$P = \{q_0 \rightarrow aq_1, q_0 \rightarrow bq_1, q_1 \rightarrow aq_1, q_1 \rightarrow bq_1, q_1 \rightarrow \epsilon\}$$



Context-Free vs. Regular Languages

Context-Free Grammars and Languages

Seen:

CFLs are more expressive than regular languages

Context-Free vs. Regular Languages

Context-Free Grammars and Languages

Seen:

CFLs are more expressive than regular languages

Open:

Decidability of word problem

The Word Problem for CFLs

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The Word Problem for CFLs

The Word Problem

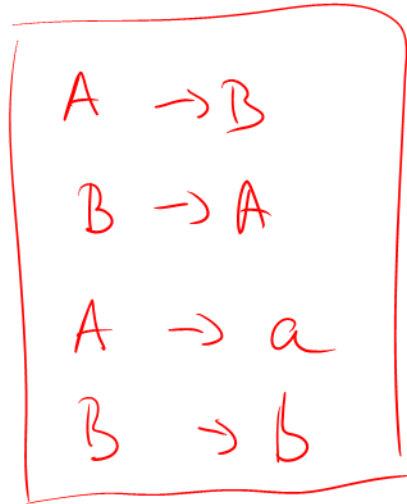
Goal: given $G = \langle N, \Sigma, P, S \rangle$ and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not

For regular languages this was easy: just let the corresponding DFA run on w .

But here: how to decide **when to stop** a derivation?

Solution: establish **normal form** for grammars which guarantees that each nonterminal produces at least one terminal symbol

$\Rightarrow$ only **finitely many combinations** to be inspected



$\Downarrow^*$

$|w| \geq n$

Chomsky Normal Form I

Definition B.8

A CFG is in **Chomsky Normal Form (Chomsky NF)** if every of its productions is of the form

$$A \rightarrow BC \quad \text{or} \quad A \rightarrow a \quad \text{or} \quad S \rightarrow \varepsilon$$

The Word Problem for CFLs

Chomsky Normal Form I

Definition B.8

A CFG is in **Chomsky Normal Form (Chomsky NF)** if every of its productions is of the form

$$\boxed{A \rightarrow BC} \quad \text{or} \quad \boxed{A \rightarrow a}$$

Example B.9

Let $S \rightarrow ab \mid aSb$ be the grammar which generates $L := \{a^n b^n \mid n \geq 1\}$.

An equivalent grammar in Chomsky NF is

$S \rightarrow AB$ $S \rightarrow AC$	$\leftarrow$	$S \rightarrow AB \mid AC$	(generates L)	$A \rightarrow$	$\boxed{B C D E}$	$\xrightarrow{G}$
		$A \rightarrow a$	(generates $\{a\}$)			$\downarrow F$
		$B \rightarrow b$	(generates $\{b\}$)			$A \rightarrow BF$
		$C \rightarrow SB$	(generates $\{a^n b^{n+1} \mid n \geq 1\}$)			$F \rightarrow CG$
						$G \rightarrow DE$

Chomsky Normal Form II

Theorem B.10

Every CFL L (with $\varepsilon \notin L$) is generatable by a CFG in Chomsky NF.

The Word Problem for CFLs

Chomsky Normal Form II

Theorem B.10

Every CFL L (with $\varepsilon \notin L$) is generatable by a CFG in Chomsky NF.

Proof.

Let L be a CFL, and let $G = \langle N, \Sigma, P, S \rangle$ be some CFG which generates L . The transformation of P into rules of the form $A \rightarrow BC$ and $A \rightarrow a$ proceeds in three steps:

1. terminal symbols only in rules of the form $A \rightarrow a$ (thus all other rules have the shape $A \rightarrow A_1 \dots A_n$)
2. elimination of “chain rules” of the form $A \rightarrow B$
3. elimination of rules of the form $A \rightarrow A_1 \dots A_n$ where $n > 2$ (details omitted)

$A \rightarrow \varepsilon$

$$\begin{array}{l} A_1 \rightarrow A_2 \quad A_2 \rightarrow A_3 \\ \dots \quad A_{n-1} \rightarrow A_n \\ \hline A_1 \rightarrow A_n \end{array}$$

$A_n B$



The Word Problem for CFLs

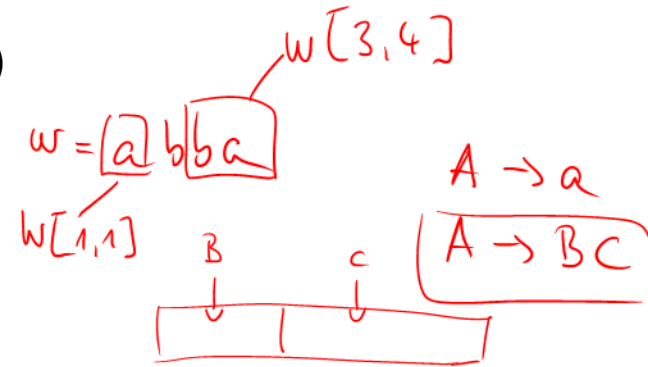
The Word Problem Revisited

Goal: given $w \in \Sigma^+$ and $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$, decide if $w \in L(G)$ or not

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

Approach by Cocke, Younger, Kasami (**CYK algorithm**):

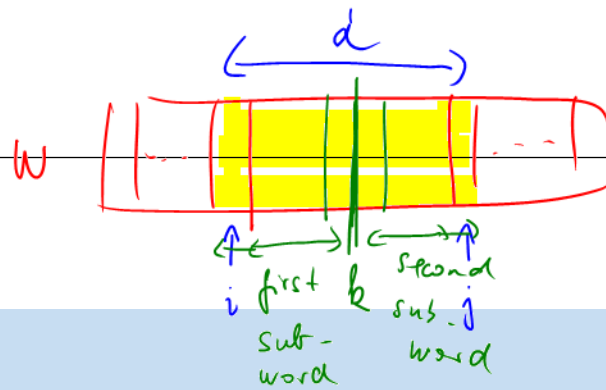
1. transform G into Chomsky NF
2. let $w = a_1 \dots a_n$ ($n \geq 1$)
3. let $w[i, j] := a_i \dots a_j$ for every $1 \leq i \leq j \leq n$
4. consider segments $w[i, j]$ in order of increasing length, starting with $w[i, i]$ (i.e., single letters)
5. in each case, determine $N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i, j]\}$ using a “dynamic programming” approach:
 - $i = j$: $N_{i,i} = \{A \in N \mid A \rightarrow w[i, i] \in P\}$
 - $i < j$: $N_{i,j} = \{A \in N \mid \exists B, C \in N, k \in \{i, \dots, j-1\} : A \rightarrow BC \in P, B \in N_{i,k}, C \in N_{k+1,j}\}$
6. test whether $S \in N_{1,n}$ (and thus, whether $S \Rightarrow^* w[1, n] = w$)



The Word Problem for CFLs

The CYK Algorithm I

Algorithm B.11 (CYK Algorithm)



Input: $G = \langle N, \Sigma, P, S \rangle$ in Chomsky NF, $w = \boxed{a_1 \dots a_n} \in \Sigma^+$

Question: $w \in L(G)$?

Procedure: for $i := 1$ to n do

$N_{i,j} := \{A \in N \mid A \rightarrow a_i \in P\}$
next i ;

for $d := 1$ to $n - 1$ do % compute $N_{i,i+d}$

for $i := 1$ to $n - d$ do

$j := i + d$; $N_{i,j} := \emptyset$;

for $k := i$ to $j - 1$ do

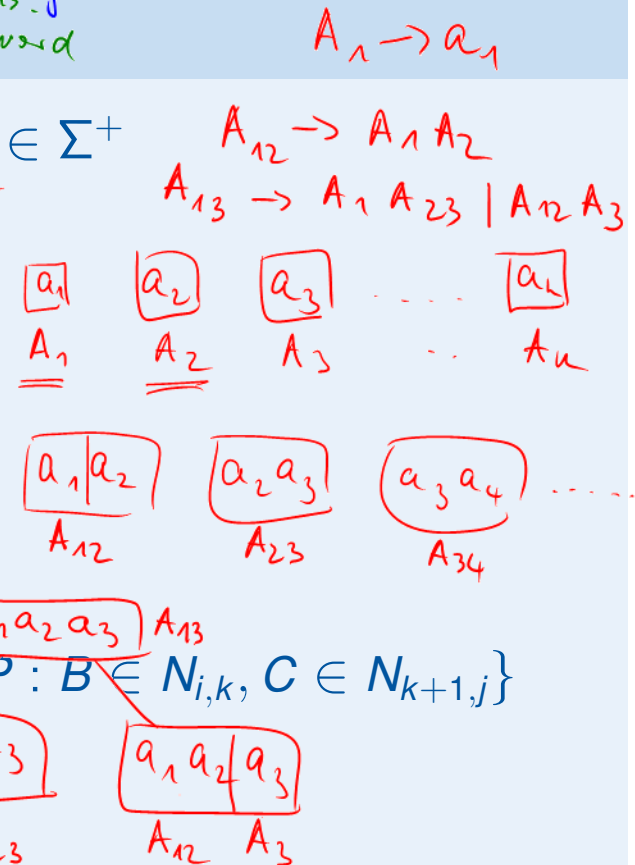
$N_{i,j} := N_{i,j} \cup \{A \in N \mid \exists A \rightarrow BC \in P : B \in N_{i,k}, C \in N_{k+1,j}\}$

next k

next i

next d

Output: “yes” if $S \in N_{1,n}$, otherwise “no”



The Word Problem for CFLs

The CYK Algorithm II

Example B.12

$G: S \rightarrow SA \mid a$

$A \rightarrow BS$

$B \rightarrow BB \mid BS \mid b \mid c$

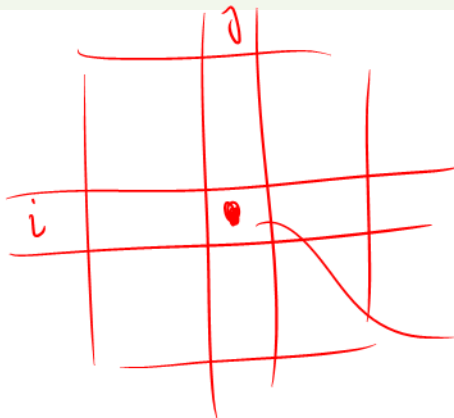
$w = abaaba$

Matrix representation of $N_{i,j}$

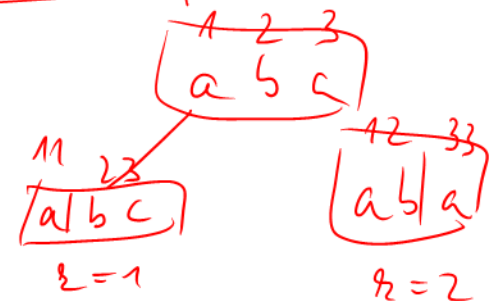
(on the board)

	1	2	3	4	5	6
1	S	-	S	S	-	S
2	-	B	A, B	A, B	B	A, B
3	-	-	S	-	-	-
4	-	-	-	S	-	S
5	-	-	-	-	B	A, B
6	-	-	-	-	-	S

$S \rightarrow$



$w = w_1 \dots [w_i \dots w_j] \dots w_n$



The Word Problem for Context-Free Languages

Seen:

Word problem decidable using CYK algorithm

The Word Problem for Context-Free Languages

Seen:

Word problem decidable using CYK algorithm

Open:

Emptiness problem

The Emptiness Problem for CFLs

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The Emptiness Problem for CFLs

The Emptiness Problem

Goal: given $G = \langle N, \Sigma, P, S \rangle$, decide whether $L(G) = \emptyset$ or not

For regular languages this was easy: check in the corresponding DFA whether some final state is reachable from the initial state.

Here: test whether start symbol is **productive**, i.e., whether it generates a terminal word

The Emptiness Problem for CFLs

The Emptiness Test

Algorithm B.13 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

Question: $L(G) = \emptyset$?

Procedure: *mark every $a \in \Sigma$ as productive;*
repeat
 if there is $A \rightarrow \alpha \in P$ such that *all symbols in α productive* then
 mark A as productive;
 end;
until *no further productive symbols found;*

Output: “no” if S productive, otherwise “yes”

The Emptiness Problem for CFLs

The Emptiness Test

Algorithm B.13 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

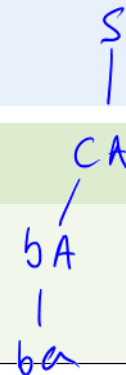
Question: $L(G) = \emptyset$?

Procedure: *mark every $a \in \Sigma$ as productive;*
 repeat
 if *there is $A \rightarrow \alpha \in P$ such that all symbols in α productive then*
 mark A as productive;
 end;
 until *no further productive symbols found;*

Output: “no” if S productive, otherwise “yes”

Example B.14

$G : \begin{array}{l|l} \boxed{S} \rightarrow AB & \boxed{CA} \\ B \rightarrow BC & AB \end{array} \quad \begin{array}{l} \boxed{A} \rightarrow \boxed{a} \\ \boxed{C} \rightarrow aB \end{array} \quad \boxed{b}$ (on the board)



The Emptiness Problem for CFLs

The Emptiness Problem for CFLs

Seen:

Emptiness problem decidable based on productivity of symbols

The Emptiness Problem for CFLs

The Emptiness Problem for CFLs

Seen:

Emptiness problem decidable based on productivity of symbols

Open:

Closure properties of CFLs

Closure Properties of CFLs

Outline of Part B

Context-Free Grammars and Languages

Context-Free vs. Regular Languages

The Word Problem for CFLs

The Emptiness Problem for CFLs

Closure Properties of CFLs

Pushdown Automata

Outlook

$$\text{op} \quad \begin{aligned} f_1 &: A \rightarrow A \\ f_2 &: A \times A \rightarrow A \end{aligned}$$

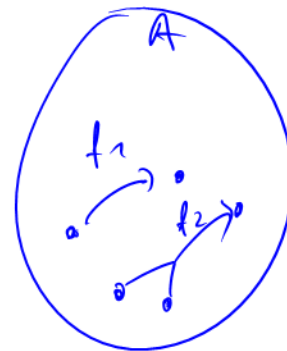
$\Rightarrow f_i$ is closed on A

$$+ : \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \rightarrow \mathbb{N}_{\geq 0}$$

$$- : \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \rightarrow \mathbb{Z}$$

$$\times : \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \rightarrow \mathbb{N}_{\geq 0}$$

$$\div : \mathbb{N}_{\geq 0} \times \mathbb{N}_{> 0} \rightarrow \mathbb{Q}$$



Regular languages $L_1, L_2 \in L_{\text{reg}} \leftarrow$ set of all regular languages

$$\cap : L_{\text{reg}} \times L_{\text{reg}} \rightarrow L_{\text{reg}}$$

L_1, L_2 regular language $\rightarrow$ DFA M_1, M_2
 $L(M_1) = L_1 \quad L(M_2) = L_2$

$$\cup : L_{\text{reg}} \times L_{\text{reg}} \rightarrow L_{\text{reg}}$$

$$\bar{} : L_{\text{reg}} \rightarrow L_{\text{reg}}$$

$$\cdot : L_{\text{reg}} \times L_{\text{reg}} \rightarrow L_{\text{reg}}$$

$$^* : L_{\text{reg}} \rightarrow L_{\text{reg}}$$

Closure Properties of CFLs

Positive Results

Theorem B.15

The set of CFLs is closed under **concatenation**, union, and iteration.

?

L^*

$L_1, L_2 \text{ CFL}$

$\downarrow$

$G_1, G_2 \quad \alpha(G_1) = L_1 \quad L(G_2) = L_2$

$G_i = (N_i, \Sigma, P_i, S_i)$

$\uparrow$
 $N_1 \cap N_2 = \emptyset$

$L = \{ w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2 \}$

$G_1: S_1 \rightarrow A_1 B_1$

$A_1 \rightarrow a$

$B_1 \rightarrow b$

$L_1 = \{ ab \}$

$G_2: S_2 \rightarrow B_2 A_2$

$A_2 \rightarrow a$

$B_2 \rightarrow b$

$L_2 = \{ ba \}$

$L_1 \cdot L_2 = \{ abba \}$

S
 $\downarrow$
 $S_1 S_2$
 $\otimes P_1$
 $ab S_2$
 $\otimes abba$

$G: \{ S \rightarrow S_1 S_2 \} \cup P_1 \cup P_2$

Closure Properties of CFLs

Positive Results

Theorem B.15

The set of CFLs is closed under concatenation, union, and iteration.

Proof.

For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$. Then

Closure Properties of CFLs

Positive Results

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$G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \rightarrow S_1 S_2\} \cup P_1 \cup P_2$ generates $L_1 \cdot L_2$;

Closure Properties of CFLs

Positive Results

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$G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \rightarrow S_1 \mid S_2\} \cup P_1 \cup P_2$ generates $L_1 \cup L_2$; and

$$G_1: \langle N_1, \Sigma, P_1, S_1 \rangle \quad P = \{S \rightarrow S_1 S \mid \epsilon\} \cup P_1 \quad \left| \quad L^* = \bigcup_{i=0}^{\infty} L^i \right.$$
$$L^i = \underbrace{L \cdot \dots \cdot L}_i$$

Closure Properties of CFLs

Positive Results

Theorem B.15

The set of CFLs is closed under concatenation, union, and iteration.

Proof.

For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$. Then

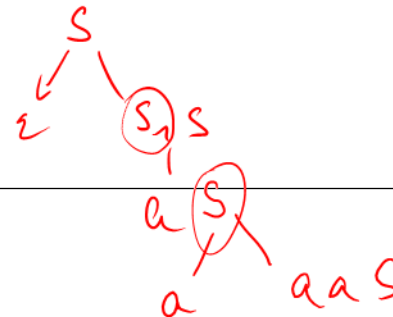
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$G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1$ and $P := \{S \rightarrow \varepsilon \mid S_1 S\} \cup P_1$ generates L_1^* .



$$\begin{aligned} S_1 &\rightarrow a \\ L_1 &= L(G_1) = \{a\} \\ L_1^* &= a^* \end{aligned}$$



Closure Properties of CFLs

Negative Results

Theorem B.16

The set of CFLs is not closed under intersection and complement.

Closure Properties of CFLs

Negative Results

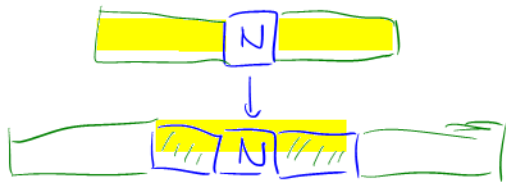
Theorem B.16

The set of CFLs is not closed under intersection and complement.

Proof.

Both $L_1 := \{a^k b^k c^l \mid k, l \in \mathbb{N}\}$ and $L_2 := \{a^k b^l c^l \mid k, l \in \mathbb{N}\}$ are CFLs, but not $L_1 \cap L_2 = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ (without proof).

$$\underline{a^k b^{k+m} c^m} \quad \checkmark \quad c \neq l$$



$$N \rightarrow a N b$$
$$N \Rightarrow a^m N b^m$$

a^n	b^n	c^n
-------	-------	-------

$\times$ not CFL

$$\boxed{N \rightarrow a N b} \quad \boxed{N \rightarrow c N} \quad N \Rightarrow a N b \Rightarrow a c N b \Rightarrow a c a N b b \Rightarrow^*$$

$(a+c)^k$	N	b^k
-----------	-----	-------

Closure Properties of CFLs

Negative Results

Theorem B.16

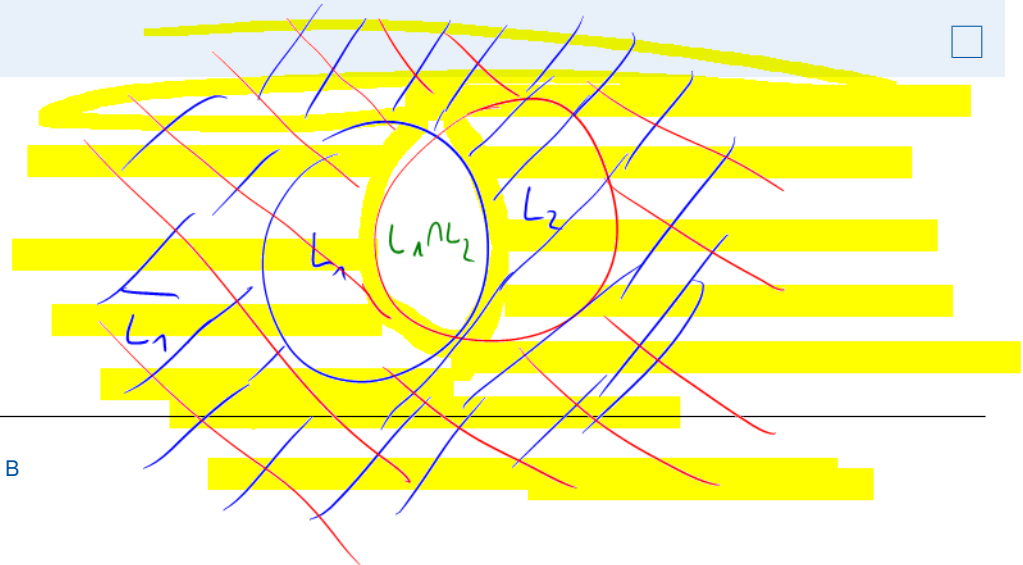
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If CFLs were closed under complement, then also under intersection (as $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$).

□



Overview of Decidability and Closure Results

Decidability Results			
Class	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+ (A.38)	+ (A.40)	+ (A.42)
CFL	+ (B.11)	+ (B.13)	–

Closure Properties of CFLs

Overview of Decidability and Closure Results

Decidability Results			
Class	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+ (A.38)	+ (A.40)	+ (A.42)
CFL	+ (B.11)	+ (B.13)	—

Closure Results					
Class	$L_1 \cdot L_2$	$L_1 \cup L_2$	$L_1 \cap L_2$	$\bar{L}$	L^*
Reg	+ (A.28)	+ (A.18)	+ (A.16)	+ (A.14)	+ (A.29)
CFL	+ (B.15)	+ (B.15)	— (B.16)	— (B.16)	+ (B.15)

Closure Properties of CFLs

Closure Properties

Seen:

Closure under concatenation, union and iteration

Non-closure under intersection and complement

Closure Properties of CFLs

Closure Properties

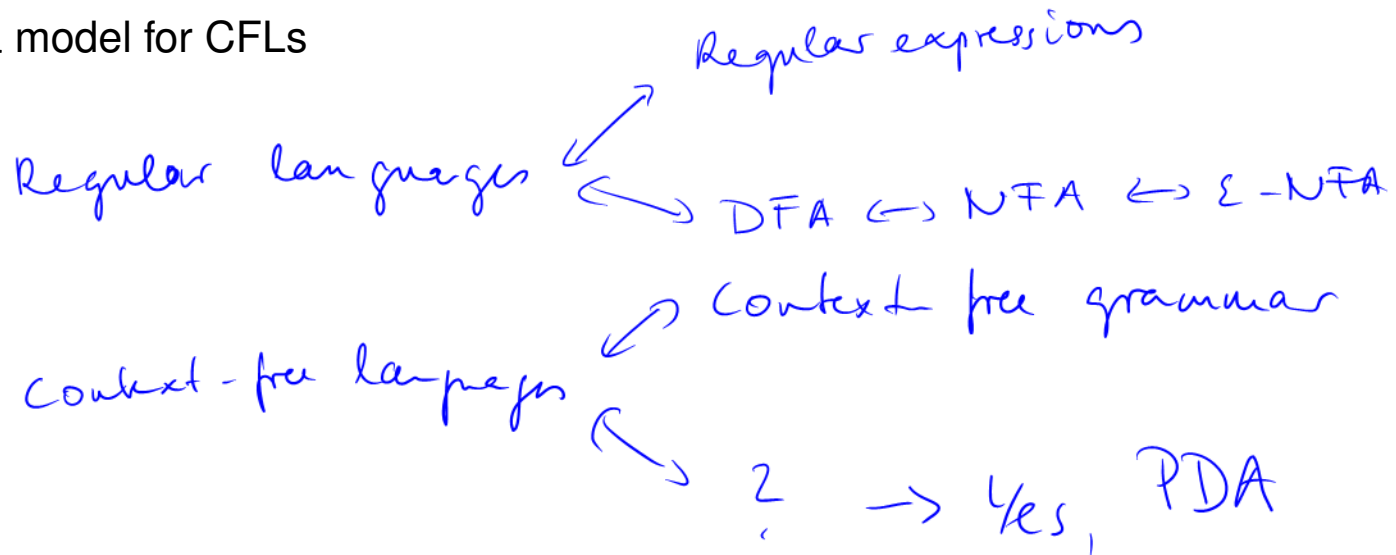
Seen:

Closure under concatenation, union and iteration

Non-closure under intersection and complement

Open:

Automata model for CFLs



Pushdown Automata

Outline of Part B

Context-Free Grammars and Languages

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Pushdown Automata

Outlook

Pushdown Automata

Pushdown Automata I

Goal: introduce an automata model which **exactly accepts CFLs**

Clear: DFA not sufficient

(missing “counting capability”, e.g. for $\{a^n b^n \mid n \geq 1\}$)

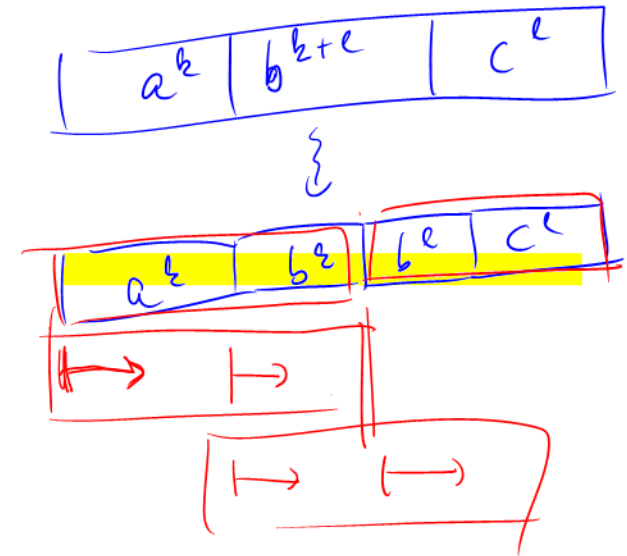
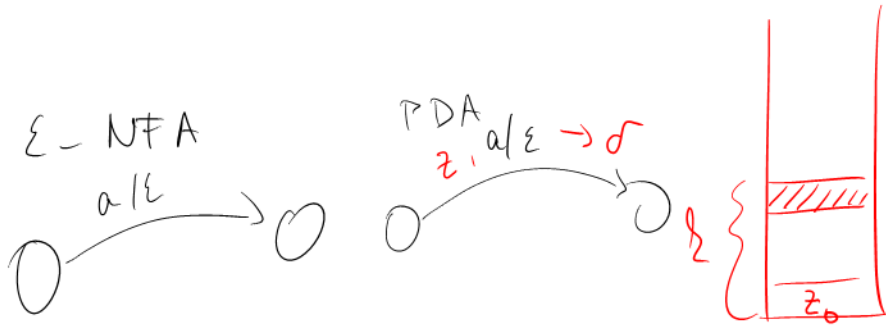
DFA will be extended to **pushdown automata** by

- adding a pushdown store which stores symbols from a pushdown alphabet and uses a special bottom symbol
- adding push and pop operations to transitions

Example for
not context-
free language.

S	N	Ma
↓	↓	↓
MN	aNb	cMad

$s \Rightarrow MN \Rightarrow MaNb \Rightarrow$
 $cMadNb \Rightarrow cMad aNb \Rightarrow$
 $\rightarrow ccMadda Nb \dots$



Pushdown Automata

$$\text{DFA: } \delta(q, a) \rightarrow q'$$
$$\text{NFA: } \Delta \subseteq Q \times \Sigma \times Q$$

Pushdown Automata II

Definition B.17

A **pushdown automaton (PDA)** is of the form $\mathcal{A} = \langle \underline{Q}, \underline{\Sigma}, \underline{\Gamma}, \underline{\Delta}, \underline{q_0}, \underline{Z_0}, \underline{F} \rangle$ where

Q is a finite set of **states**

Σ is the (finite) **input alphabet**

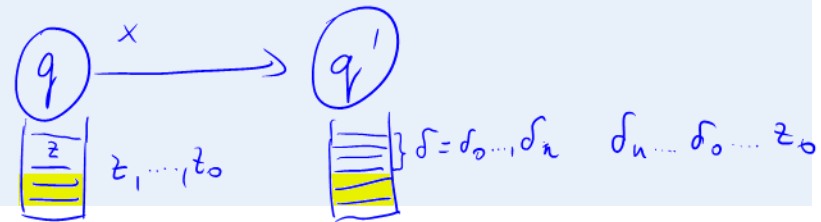
Γ is the (finite) **pushdown alphabet**

$\Delta \subseteq (Q \times \Gamma \times \Sigma_\epsilon) \times (Q \times \Gamma^*)$ is a finite set of **transitions**

$q_0 \in Q$ is the **initial state**

Z_0 is the **(pushdown) bottom symbol**

$F \subseteq Q$ is a set of **final states**



Interpretation of $((q, Z, x), (q', \delta)) \in \Delta$: if the PDA $\mathcal{A}$ is in state q where Z is on top of the stack and x is the next input symbol (or empty), then $\mathcal{A}$ reads x , replaces Z by δ , and changes into the state q' .

Pushdown Automata

Configurations, Runs, Acceptance

Definition B.18

Let $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ be a PDA.

An element of $Q \times \Gamma^* \times \Sigma^*$ is called a **configuration** of $\mathcal{A}$.

The **initial configuration** for input $w \in \Sigma^*$ is given by (q_0, Z_0, w) .

The set of **final configurations** is given by $F \times \{\varepsilon\} \times \{\varepsilon\}$.

If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.

$$\begin{array}{l} q \xrightarrow{x} q' \quad (q, Z\gamma, xw) \rightarrow (q', \delta\gamma, w) \\ (q, xw) \rightarrow (q', w) \\ \text{Operational Semantics} \end{array}$$

$$\frac{q, q' \in Q \quad x \in \Sigma \quad w \in \Sigma^* \quad \delta(q, x) = q'}{q \xrightarrow{x} q'} \text{ Rule}$$

$$0 \cdot x = 0 \quad \text{Axiom}$$

$$\begin{array}{l} \text{Semantics : } q \in R_{\mathcal{A}}(w) \models q' \in R_{\mathcal{A}}(wa) \\ \delta(q, a) = q' \\ \text{Proof system : } \vdash \end{array}$$

Configurations, Runs, Acceptance

Definition B.18

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If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.

$\mathcal{A}$ **accepts** $w \in \Sigma^*$ if $(q_0, Z_0, w) \vdash^* (q, \varepsilon, \varepsilon)$ for some $q \in F$.

The **language accepted by** $\mathcal{A}$ is $L(\mathcal{A}) := \{w \in \Sigma^* \mid \mathcal{A} \text{ accepts } w\}$.

A language L is called **PDA-recognisable** if $L = L(\mathcal{A})$ for some PDA $\mathcal{A}$.

Two PDA $\mathcal{A}_1, \mathcal{A}_2$ are called **equivalent** if $L(\mathcal{A}_1) = L(\mathcal{A}_2)$.

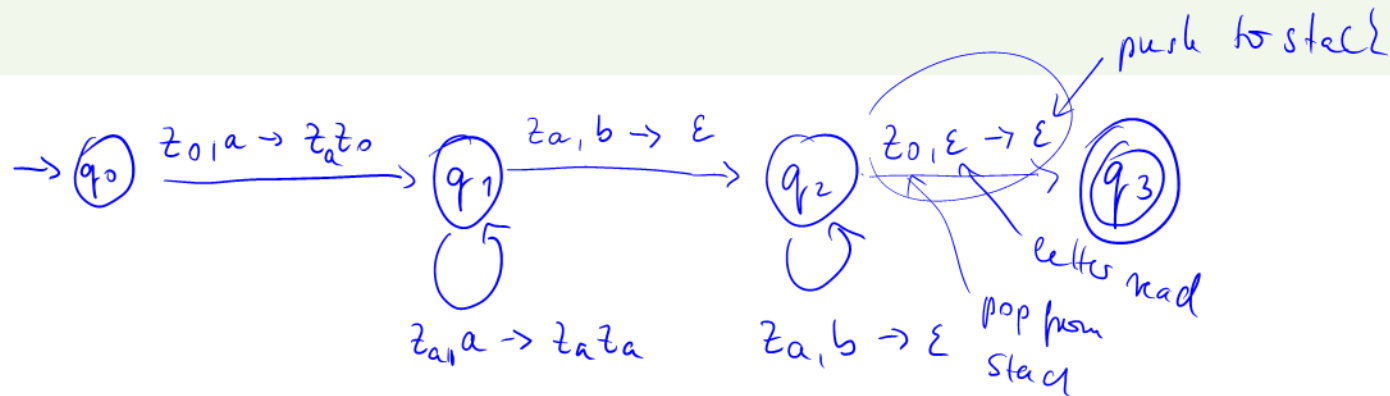
Pushdown Automata

Examples

Example B.19

1. PDA which recognises $L = \{a^n b^n \mid n \geq 1\}$
(on the board)

Wanted: $M = \langle Q, \Sigma, \Gamma, \Delta, q_0, z_0, \# \rangle$ that recognizes L



Examples

Example B.19

1. PDA which recognises $L = \{a^n b^n \mid n \geq 1\}$
(on the board)
2. PDA which recognises $L = \{ww^R \mid w \in \{a, b\}^*\}$
(**palindromes** of even length; on the board)

Observation: $\mathcal{P}_2$ is nondeterministic: whenever a construction transition is applicable, the pushdown could also be deconstructed

Deterministic PDA

Definition B.20

A PDA $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called **deterministic (DPDA)** if for every $q \in Q, Z \in \Gamma$,

1. for every $x \in \Sigma_\epsilon$, there is at most one (q, Z, x) -transition in Δ and
2. if there is a (q, Z, a) -transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ϵ) -transition in Δ .

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
 $(q'_1, \delta_1 \gamma, w) \vdash (q, Z \gamma, xw) \vdash (q'_2, \delta_2 \gamma, w)$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \epsilon), (q'_2, \delta_2)) \in \Delta$:
 $(q'_1, \delta_1 \gamma, w) \vdash (q, Z \gamma, aw) \vdash (q'_2, \delta_2 \gamma, aw)$

Pushdown Automata

$$\Delta \subseteq (Q \times \Gamma \times \Sigma_\epsilon) \times (Q \times \Gamma^*)$$

$$(q, \Gamma, a) \rightarrow (q', \delta)$$

$$\rightarrow (q'', \delta')$$

$$(q, \Gamma, \epsilon) \rightarrow (q''', \delta''')$$

for all $q \in Q, \Gamma \in \Gamma$

Deterministic PDA

Definition B.20

$$\Delta : (Q \times \Gamma \times \Sigma_\epsilon) \rightarrow (Q \times \Gamma^*) \text{ s.t.}$$

A PDA $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called **deterministic (DPDA)** if for every $q \in Q, Z \in \Gamma$,

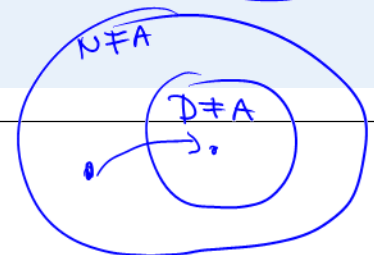
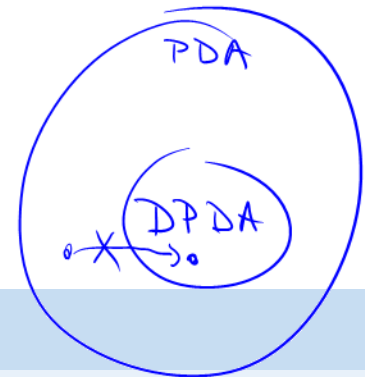
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 $(q'_1, \delta_1 \gamma, w) \vdash (q, Z \gamma, aw) \vdash (q'_2, \delta_2 \gamma, aw)$

Corollary B.21

In a DPDA, every configuration has at most one $\vdash$ -successor.



Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages
(DPDA-recognisable languages are **closed under complement**, which is generally not true for PDA-recognisable languages)

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognisable languages are **closed under complement**, which is generally not true for PDA-recognisable languages)

Example B.22

The set of palindromes of even length is PDA-recognisable, but not DPDA-recognisable (without proof).

PDA and Context-Free Languages I

Theorem B.23

A language is context-free iff it is PDA-recognisable.

Pushdown Automata

PDA and Context-Free Languages I

Theorem B.23

A language is context-free iff it is PDA-recognisable.

*L is context-free language
↓
exist context-free grammar
 G s.t. $\mathcal{L}(G) = L$*

Proof.

$\Leftarrow$: omitted

$\Rightarrow$: let $G = \langle N, \Sigma, P, S \rangle$ be a CFG. Construction of PDA $\mathcal{A}_G$ recognising $L(G)$:

$\mathcal{A}_G$ simulates a derivation of G where always the leftmost nonterminal of a sentence is replaced ("leftmost derivation")

begin with S on pushdown

if nonterminal on top: apply a corresponding production rule

if terminal on top: match with next input symbol

(cf. formal construction on following slide)

*$w = \dots N_1 \dots N_2 \dots$
=*

*$z_0 \equiv S$
 $S \rightarrow \sigma$
 $S, \epsilon \rightarrow \sigma$
 $(q_1, \begin{bmatrix} \bar{a} \\ S \end{bmatrix}, a_w) \xrightarrow{a, a \rightarrow \epsilon(q_1)} (q_1, \begin{bmatrix} \bar{a} \\ \sigma \end{bmatrix}, w)$
 $\sim a w$*

□

PDA and Context-Free Languages II

$$G = (N, \Sigma, \underline{P}, S)$$

Proof of Theorem B.23 (continued).

$\Rightarrow$: Formally: $\mathcal{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

$$Q := \{q_0\}$$

$$\Gamma := N \cup \Sigma$$

① for each $A \rightarrow \alpha \in P$: $((q_0, \underline{A}, \varepsilon), (q_0, \alpha)) \in \Delta$

② for each $a \in \Sigma$: $((q_0, \underline{a}, \underline{a}), (q_0, \varepsilon)) \in \Delta$

$$Z_0 := S$$

$$F := Q$$



$$w \in (N \cup \Sigma)^*$$



Pushdown Automata

PDA and Context-Free Languages II

Proof of Theorem B.23 (continued).

$\Rightarrow$: Formally: $\mathcal{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

$$Q := \{q_0\}$$

$$\Gamma := N \cup \Sigma$$

for each $A \rightarrow \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$

for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$

$$Z_0 := S$$

$$F := Q$$



Example B.24

“Bracket language”, given by G :

$$S \rightarrow \langle \rangle \mid \langle S \rangle \mid SS$$

(on the board)



CFG: $S \rightarrow \langle \rangle \mid \langle s \rangle \mid SS$

$N = \{S\}$

$\Sigma = \{\langle, \rangle\}$

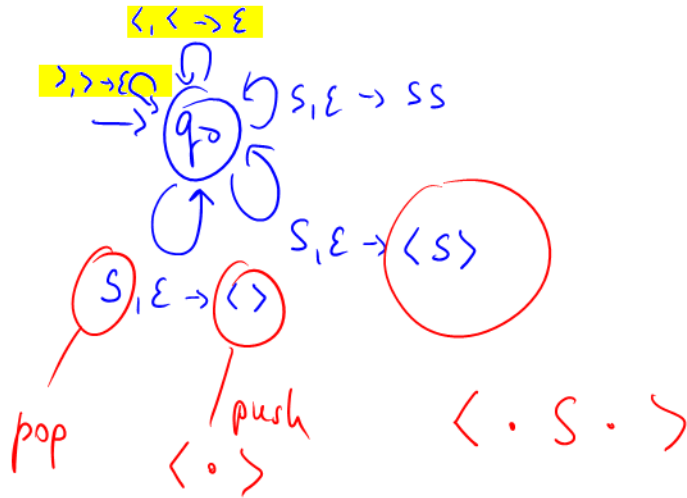
PDA: $Q = \{q_0\}$

$\Gamma = \{\langle, \rangle, S\}$

$\Sigma = \{\langle, \rangle\}$

$\Sigma_0 = S$

$F = \{q_0\}$



Derivation steps:

$$\begin{aligned}
 (q_0, S, \langle \langle \rangle \langle \rangle \rangle) &\rightarrow (q_0, \langle S \rangle, \langle \langle \rangle \langle \rangle \rangle) \rightarrow (q_0, SS, \langle \rangle \langle \rangle \rangle) \rightarrow (q_0, SS, \langle \rangle \langle \rangle \rangle) \\
 &\rightarrow (q_0, \langle \rangle SS, \langle \rangle \langle \rangle \rangle) \rightarrow (q_0, \rangle SS, \rangle \langle \rangle \rangle) \\
 &\rightarrow (q_0, S \rangle, \langle \rangle \rangle) \rightarrow (q_0, \langle \rangle \rangle, \langle \rangle \rangle) \\
 &\rightarrow (q_0, \rangle \rangle, \rangle \rangle) \rightarrow (q_0, \rangle, \rangle) \rightarrow (q_0, \epsilon, \epsilon)
 \end{aligned}$$

Stack configuration:

$\langle$
S
$\rangle$

Final steps:

$$\begin{aligned}
 (q_0, S, \rangle) &\rightarrow (q_0, \langle \rangle, \rangle) \\
 &\rightarrow (q_0, \langle SS, \rangle)
 \end{aligned}$$

Outlook

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Outlook

$$\begin{bmatrix} a & 1 \\ 1 & 1 \\ 1 & a \end{bmatrix}$$