

Exercises (Regular Languages)

1 Finite Automata

1.1 Exercise: Construct a DFA over $\Sigma := \{a, b\}$ that accepts the following language:

$$\{w \in \Sigma^* \mid \text{each } a \text{ followed by exactly 1 or 3 } b\text{'s}\}$$

directly

... ab...
... abbb...

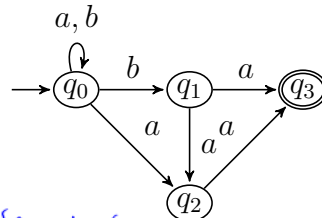
1.2 Exercise: Construct a DFA over $\Sigma := \{0, 1\}$ that accepts the following language:

$$\{w \in \Sigma^* \mid \text{decimal value of } w \text{ divisible by 4}\}$$

$$\begin{array}{r|l} 21 & 20 \\ \hline 10 & 1 \\ \hline 100 & \end{array} = 5 \quad \times$$

$$\begin{array}{r|l} 10 & 0 \\ \hline 100 & \end{array} = 4 \quad \checkmark$$

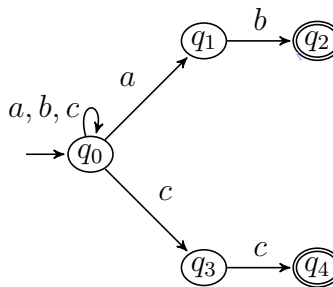
1.3 Exercise: Let $\mathfrak{A}$ be the following NFA over $\Sigma := \{a, b\}$.



$\{q_0\}$ $\{q_0, q_1\}$ $\{q_0, q_2, q_1\}$ $\{q_0, q_2, q_3\}$

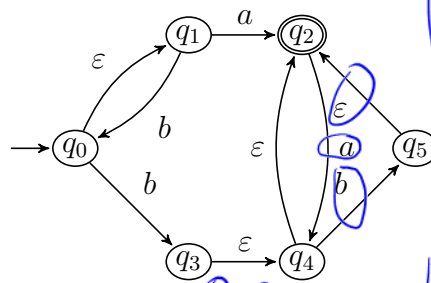
Determine the reachability sets $R_{\mathfrak{A}}(\varepsilon)$, $R_{\mathfrak{A}}(b)$, $R_{\mathfrak{A}}(ba)$, and $R_{\mathfrak{A}}(baa)$.

1.4 Exercise: Apply the powerset construction to transform the following NFA $\mathfrak{A}$ over $\Sigma := \{a, b, c\}$ into an equivalent DFA.



$\rightarrow \{q_0\} \xrightarrow{a} \{ \dots \}$
 $\rightarrow \{q_0\} \xrightarrow{b} \{ \dots \}$
 $\rightarrow \{q_0\} \xrightarrow{c} \{ \dots \}$

1.5 Exercise: Eliminate all ε -transitions of the following ε -NFA $\mathfrak{A}$ over $\Sigma := \{a, b\}$ to obtain an equivalent NFA.



if $q_0 \xrightarrow{\varepsilon^*} q \in F$
then $F' = F \cup \{q_0\}$
else $F' = F$

② $q'_0 = q_0$

③ $(p, \varepsilon, q) \in \Delta \rightarrow \text{remove it}$

④ $\begin{array}{c} \xrightarrow{\varepsilon^*} \xrightarrow{a} \xrightarrow{\varepsilon^*} \\ p \quad q \quad r \end{array}$
 $(p, q, r) \in \Delta'$

$(Q, \Sigma, \Delta, q_0, F) \rightarrow (Q, \Sigma, \Delta', q'_0, F')$

$$\Delta' = \underbrace{\left(\Delta \setminus \{ (p, \varepsilon, q) \mid p, q \in Q \} \right)}_A \cup \underbrace{\{ (p, q, r) \mid (p, \varepsilon, q) \in \Delta, (q, a, r) \in \Delta \}}_B$$

Do these statements hold or not?

1.6 a) $K \cdot L = L \cdot K$ $\times$ in general, for all K, L_1, L_2 ?
 b) $(K \cup L)^* \stackrel{?}{=} (K^*) \cup (L^*)$ $\times$

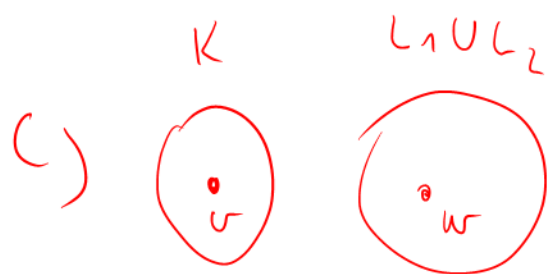
c) $K \cdot (L_1 \cup L_2) = (K \cdot L_1) \cup (K \cdot L_2)$ $\checkmark$

d) $K \cdot L \not\subseteq K \cdot L^2$ $\times$ $L = \{a\}$
 $K = \{b\}$

e) $\epsilon \in L^*$ $\checkmark$

f) $L \subseteq L^*$ $\checkmark$ $L^* = \bigcup_{n=0}^{\infty} L^n$

g) $L_1 \circ (L_1 \circ L_2) = (L_1 \circ L_2) \circ L_3$ $\checkmark$



$a \cdot (b \cdot c) = (a \cdot b) \cdot c =$

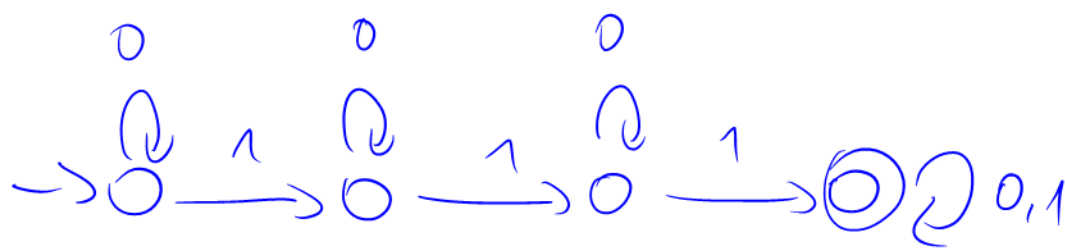
$a \cdot b \cdot c$

$K \cdot (L_1 \cup L_2) = \{vw \mid v \in K \wedge w \in L_1 \cup L_2\} =$
 $\{vw \mid v \in K \wedge (w \in L_1 \vee w \in L_2)\} =$
 $\{vw \mid v \in K \wedge w \in L_1\} \cup \{vw \mid v \in K \wedge w \in L_2\} =$
 $K \cdot L_1 \cup K \cdot L_2$

Please define DFA $\mathcal{A}$ over $\mathbb{B} = \{0, 1\}$

1.7

$$L(\mathcal{A}) = \{w \in \mathbb{B}^* \mid w \text{ has at least 3 "1"s}\}$$



"no 1"

"one 1"

"two 1s"

"at least three 1s"

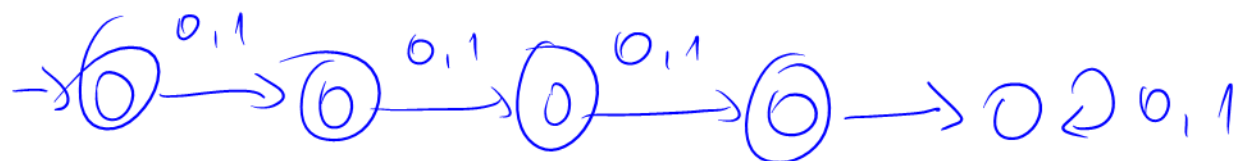
1.8

Please define DFA for $\mathbb{B} = \{0, 1\}$

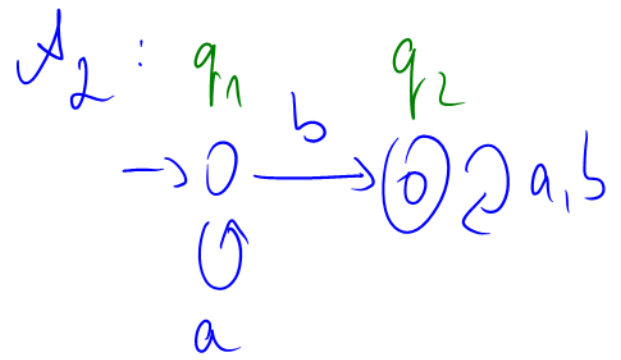
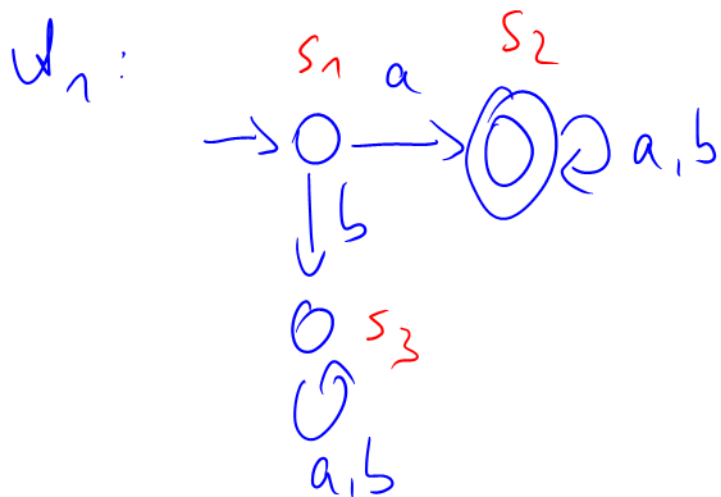
$$L(\mathcal{A}) = \{w \in \mathbb{B}^* \mid |w| \leq 3\}$$

$$L^* = \bigcup_{n=0}^{\infty} L^n = \{\epsilon\} \cup L \cup L \cdot L \cup L \cdot L \cdot L \cup \dots$$

$$\mathbb{B}^* = \bigcup_{n=0}^{\infty} \mathbb{B}^n = \{\epsilon\} \cup \mathbb{B} \cup \mathbb{B} \cdot \mathbb{B} \cup \mathbb{B} \cdot \mathbb{B} \cdot \mathbb{B} \cup \dots$$



1.3 Assume two DFA over $\Sigma = \{a, b\}$

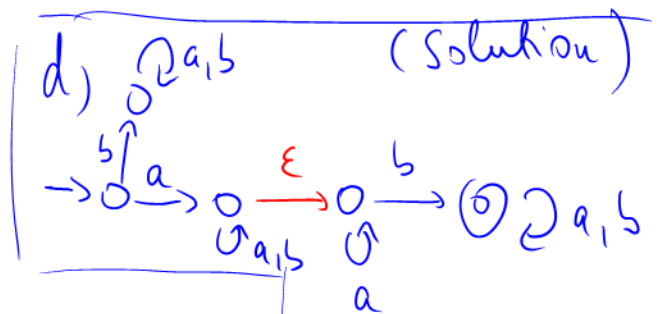


a) $\mathcal{L}(M) = \overline{\mathcal{L}(M_1)}$

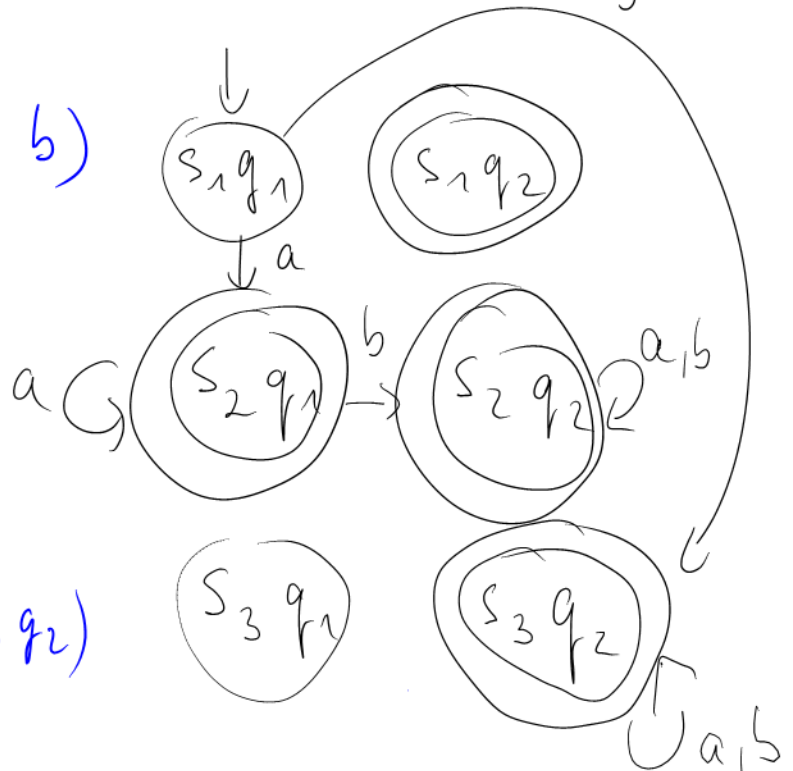
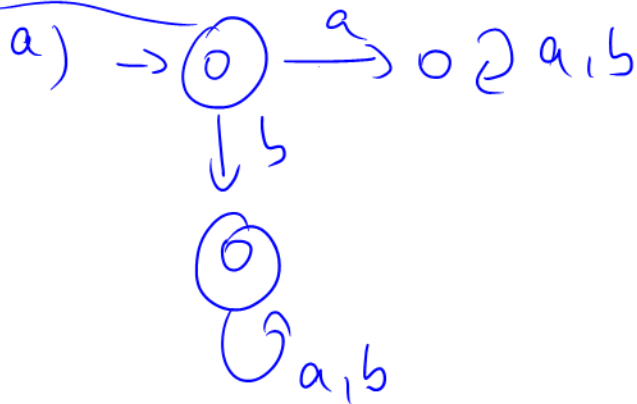
b) $\mathcal{L}(M) = \mathcal{L}(M_1) \cup \mathcal{L}(M_2)$

c) $\mathcal{L}(M) = \mathcal{L}(M_1) \cap \mathcal{L}(M_2)$

d) $\mathcal{L}(M) = \mathcal{L}(M_1) \cdot \mathcal{L}(M_2)$

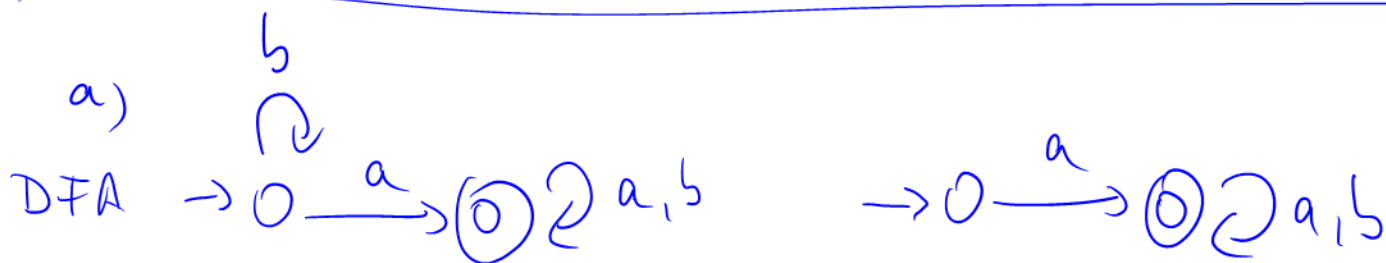


Solution:

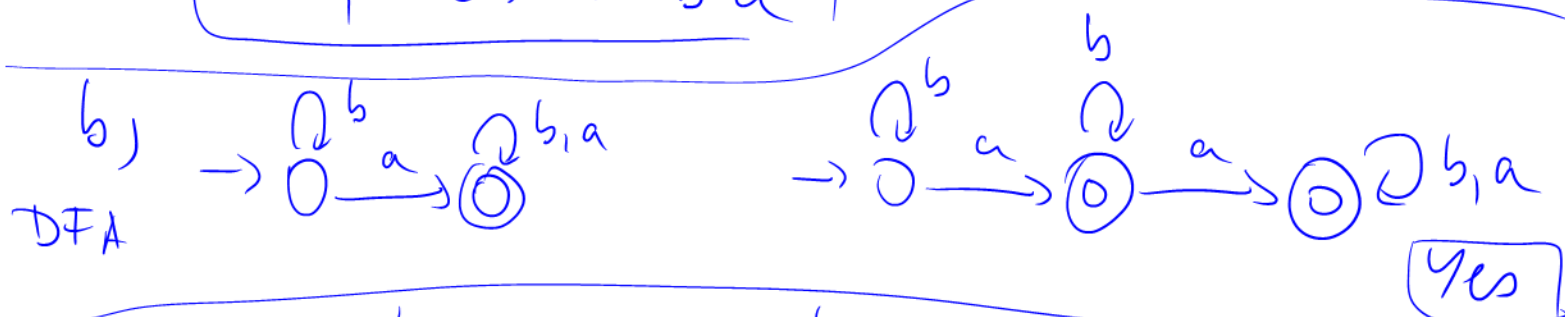


c) As b) but only (s_2, q_2) is accepting.

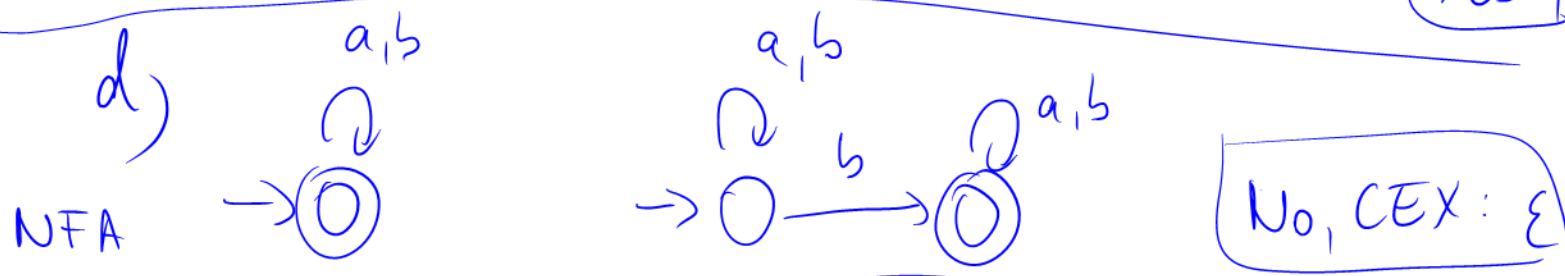
1.10 Are these automata equivalent?



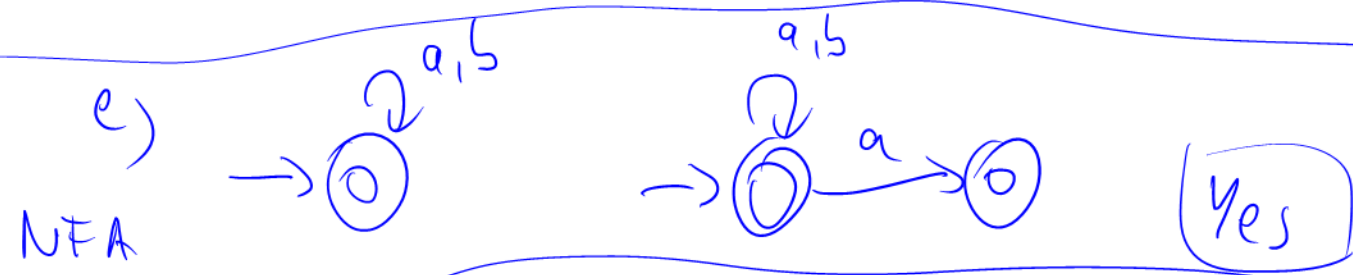
No, CEX: ba



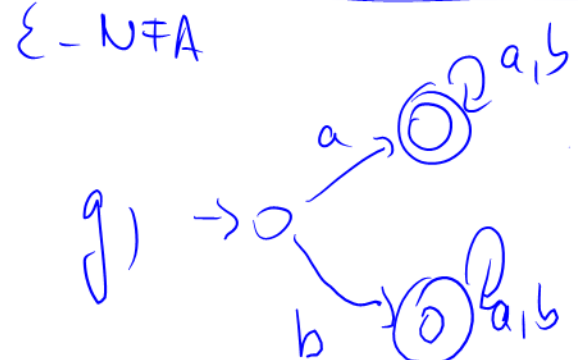
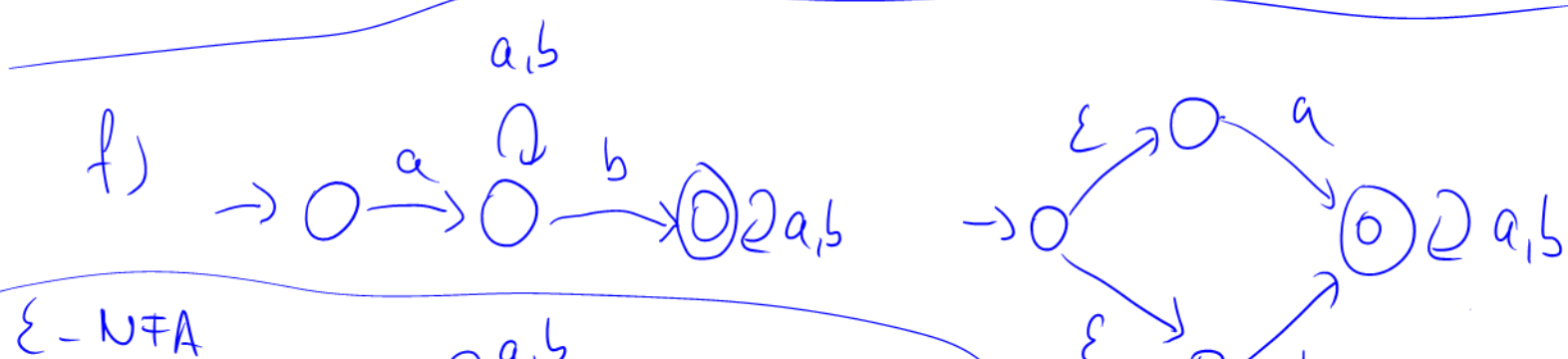
Yes



No, CEX: ϵ



Yes

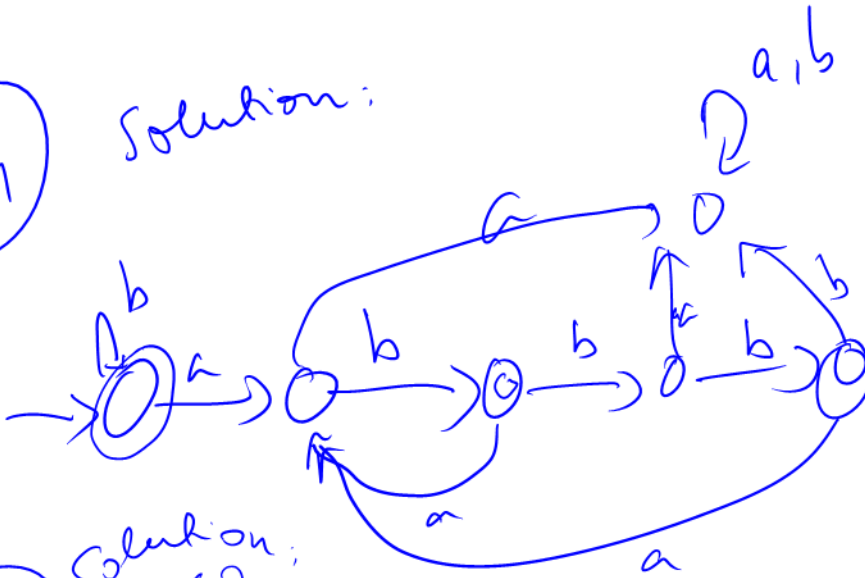


No, CEX: a

No, CEX: ϵ

1.1

Solution:

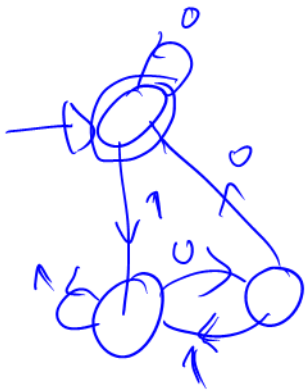


1.2

Solution:

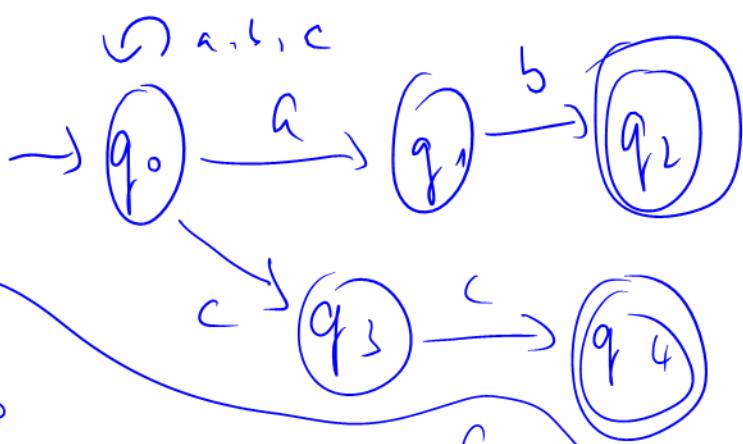


← if least significant bit would be on the left

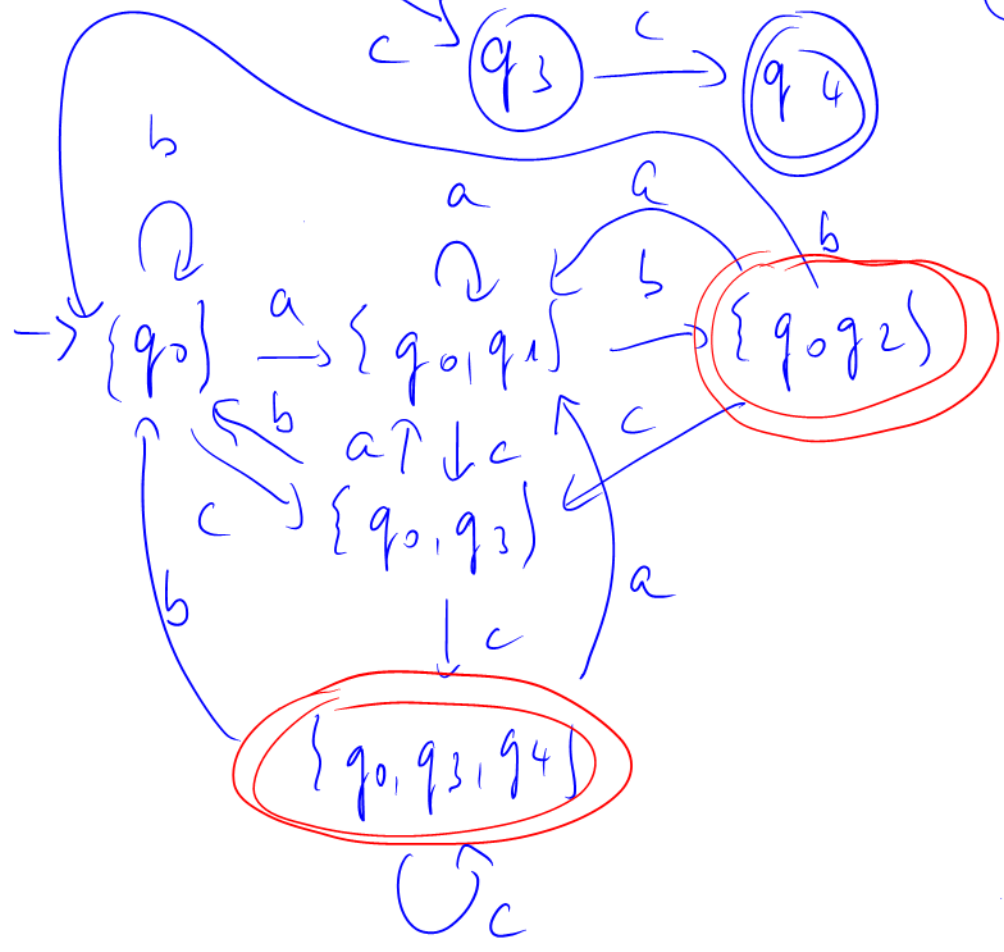


← least significant bit is on the right

1.4

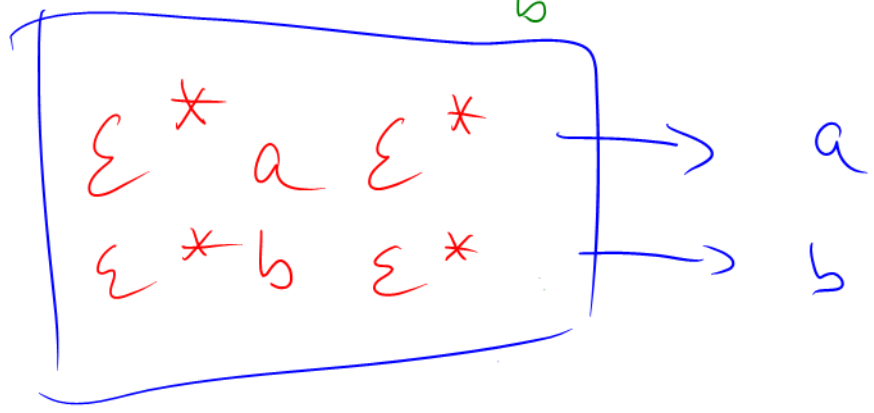
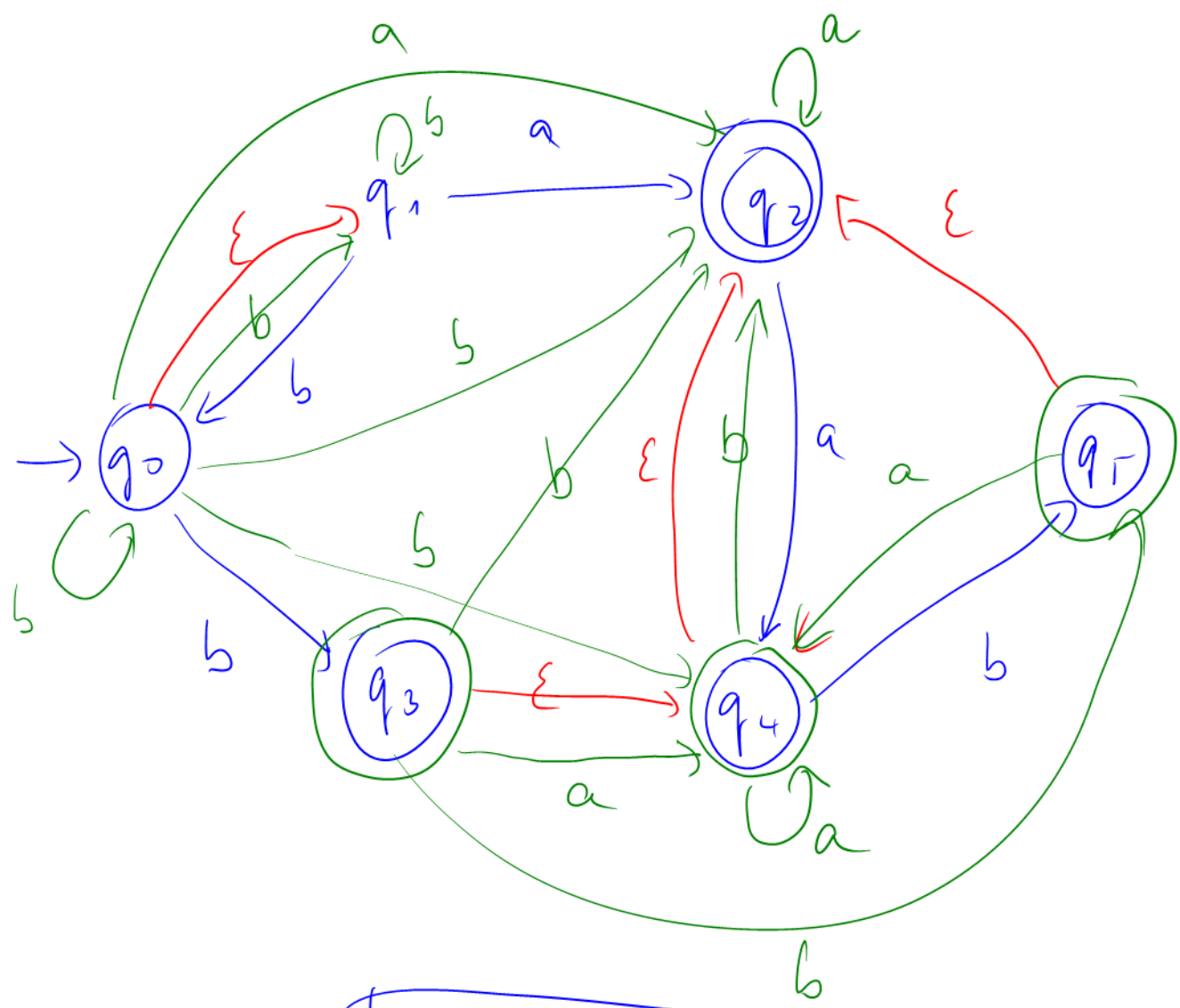


$$(a|b|c)^*(ab|cc)$$



1.5

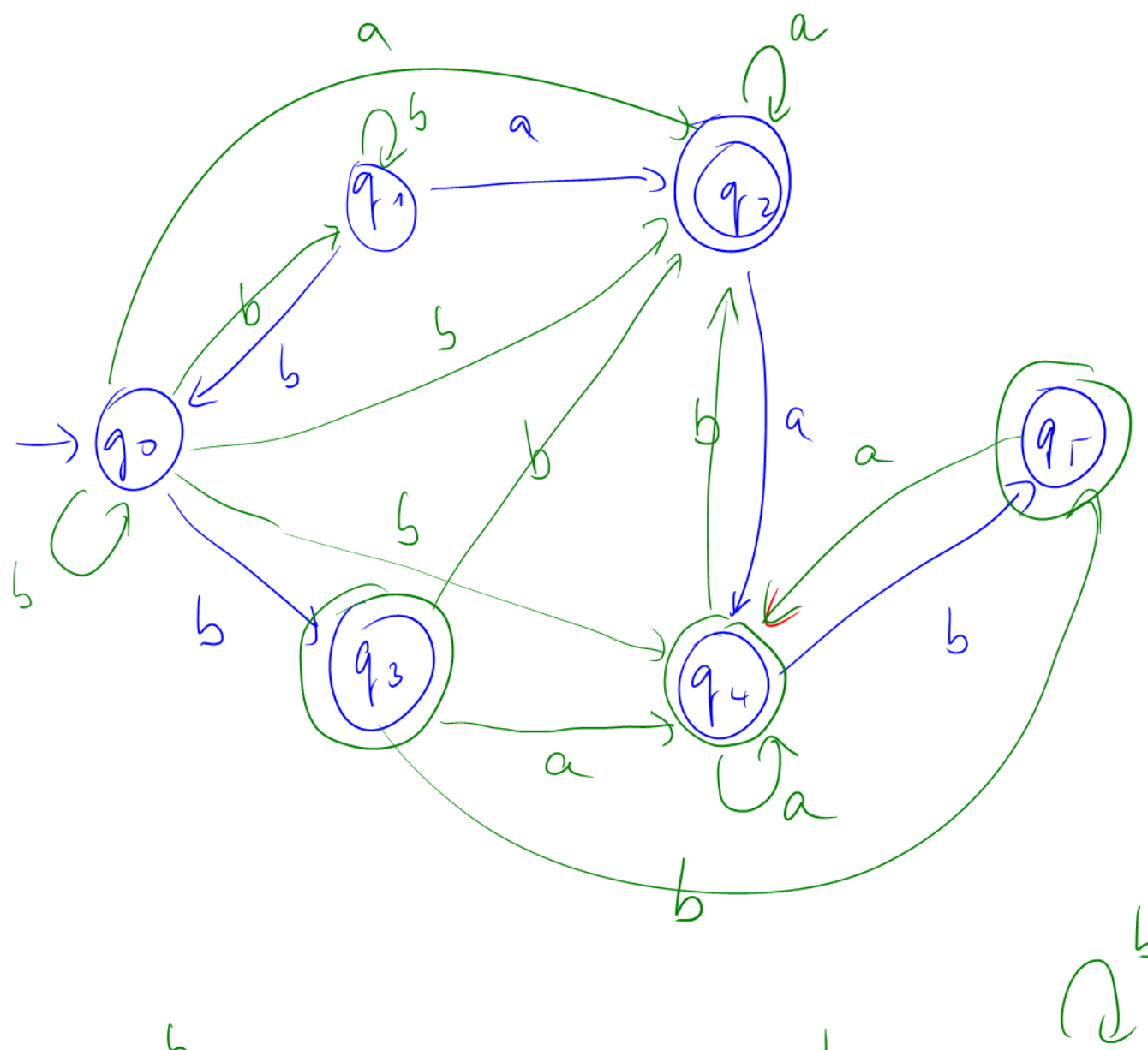
$$\Sigma = \{a, b\}$$



1.5

$p \xrightarrow{w} F$

$q \xrightarrow{\tilde{w}} F$

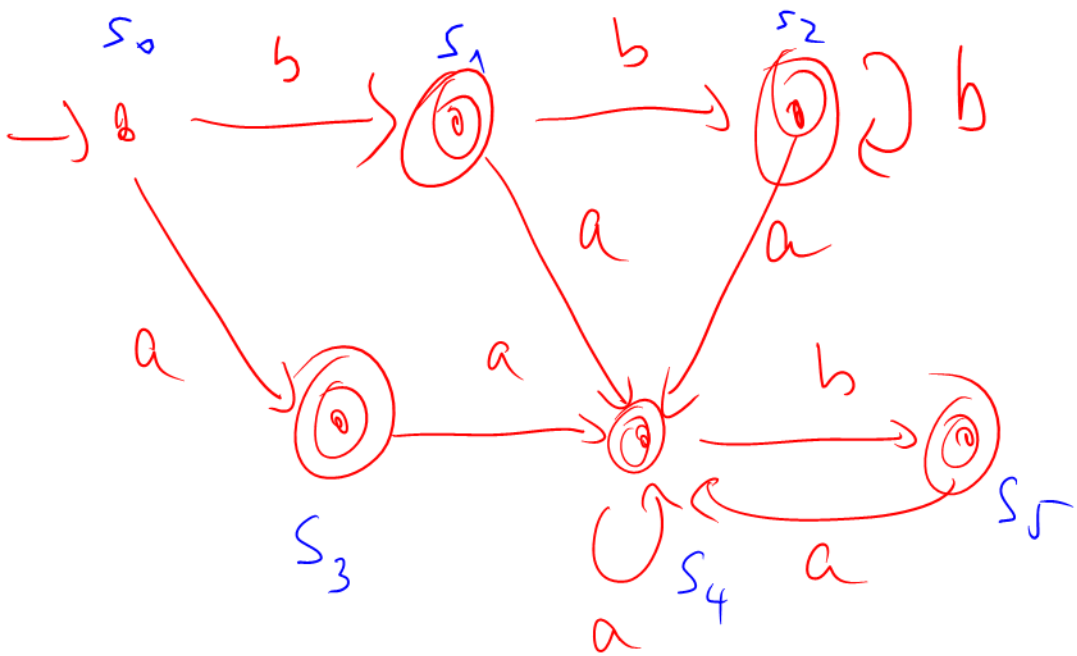


$\rightarrow \{q_0\} \xrightarrow{b} \{q_0, q_1, q_2, q_3, q_4\} \xrightarrow{b} \{q_0, q_1, q_2, q_3, q_4, q_5\}$

$\swarrow a \quad \downarrow a \quad \swarrow b \quad \searrow a$

$\{q_2\} \xrightarrow{a} \{q_2, q_4\} \xrightarrow{a} \{q_2, q_5\}$

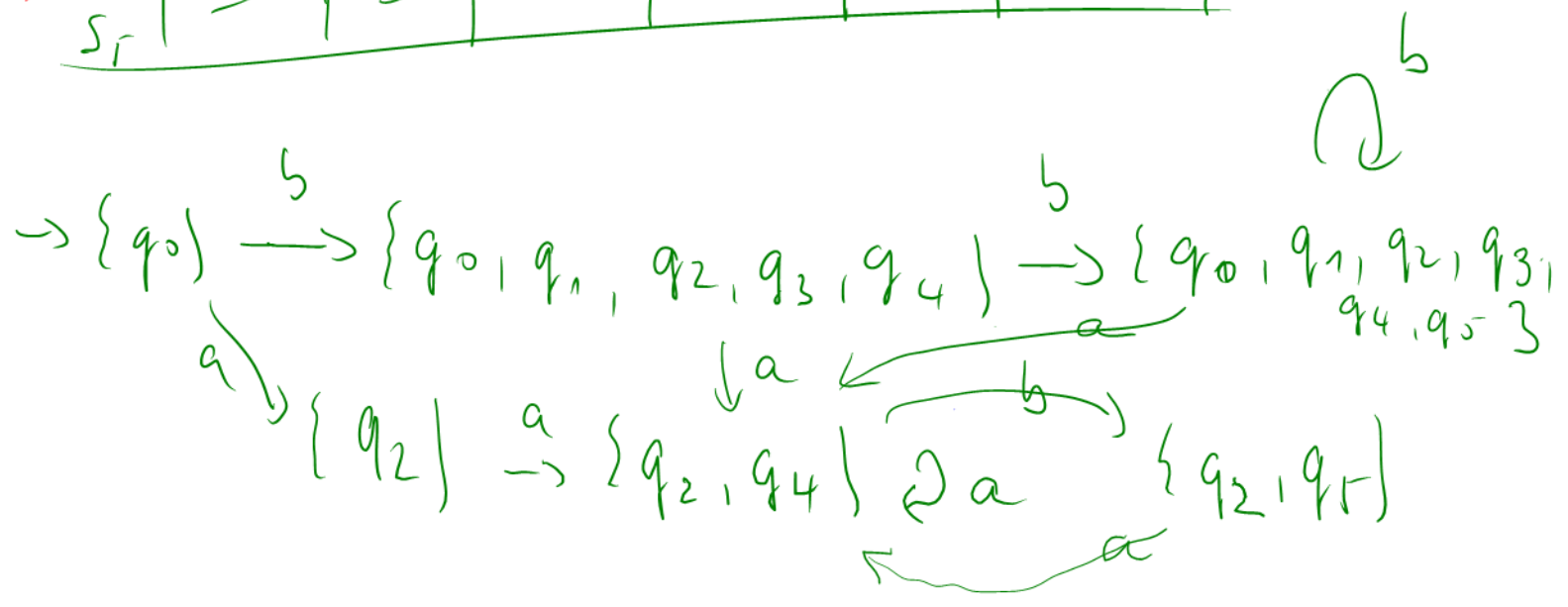
(15)

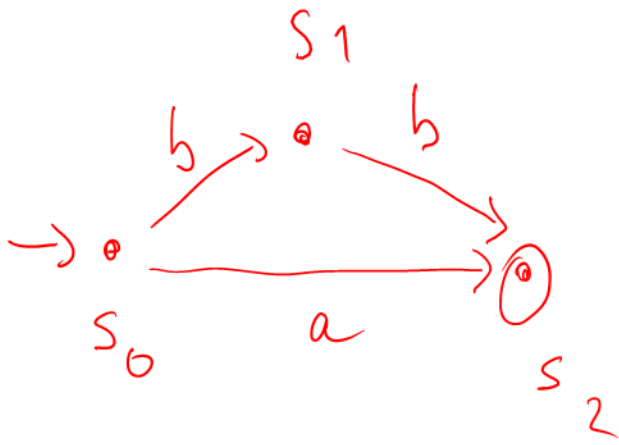


$s_i \xrightarrow{a} s_i' \in \text{col}$
 $s_j \xrightarrow{a} s_j' \in \text{row}$

	s_0	s_1	s_2	s_3	s_4	s_5
s_0	-	ϵ	ϵ	ϵ	ϵ	ϵ
s_1	-	-				
s_2	-	-	-			
s_3	-	-	-	-		
s_4	-	-	-	-	-	
s_5	-	-	-	-	-	-

w



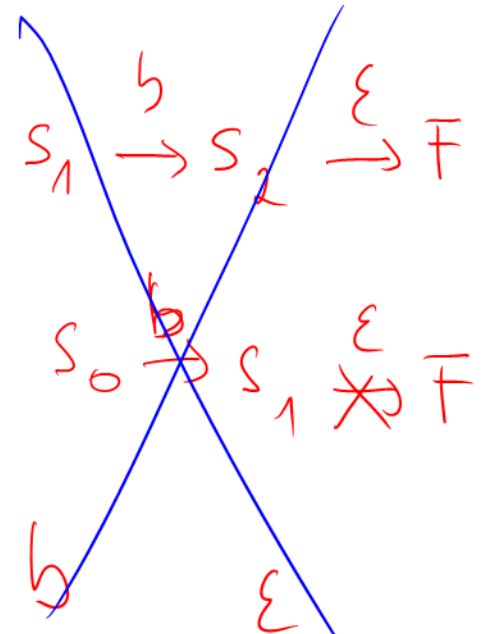
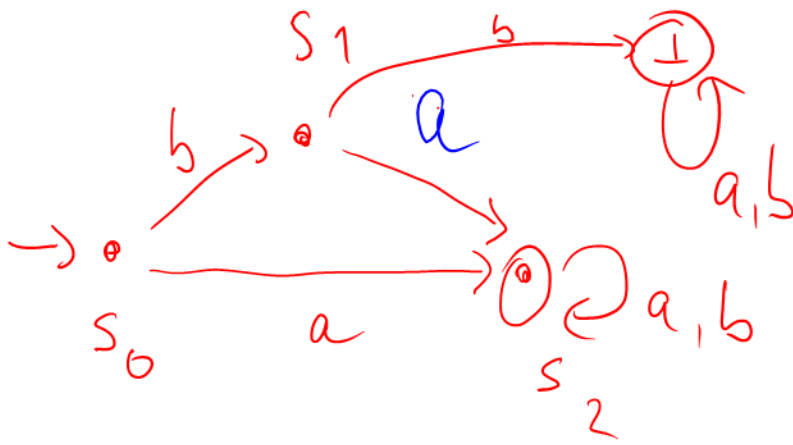


$$s_1 \xrightarrow{b} s_2 \xrightarrow{\epsilon} \bar{F}$$

$$s_0 \xrightarrow{b} s_1 \xrightarrow{\epsilon} \bar{F}$$

b ε

	s_0	s_1	s_2
s_0	—	b	ε
s_1	—	—	ε
s_2	—	—	—



	s_0	s_1	s_2	1
s_0	—	ba	ϵ	a
s_1	—	—	ϵ	a
s_2	—	—	—	ϵ
1	—	—	—	—

$s_0 \xrightarrow{b} s_1 \xrightarrow{a} s_2$
 $s_1 \xrightarrow{b} 1 \xrightarrow{a} 1$
 $ba \quad a \quad \epsilon$

$s_0 \xrightarrow{a} s_2$
 $1 \xrightarrow{a} 1$
 $a \quad \epsilon$

$s_2 \xrightarrow{b} s_2$
 $s_0 \xrightarrow{b} s_1$
 ϵ

$\xrightarrow{a} s_2$
 s_0
 ϵ

2 Regular Expressions

1.11

Exercise: Give regular expressions that describe the following languages.

1. $L := \{w \in \{a, b\}^* \mid |w| \text{ divisible by } 3\}$

$$(aaa + aab + \dots + bbb)^* \text{ or } ((a+b)(a+b)(a+b))^*$$

2. $L := \{w \in \{a, b, c\}^* \mid w \text{ does not contain } a, b, \text{ or } c\}$

$$(a+b)^* + (b+c)^* + (a+c)^*$$

3. $L := \{w \in \{a, b\}^* \mid \text{substring } ab \text{ occurs exactly twice in } w, \text{ but not at the end}\}$

$$b^* a^+ b^+ a^+ b^+ (a^+ b^+)^*$$

1.12

Exercise: Show that regular languages are closed under the reversal operation.

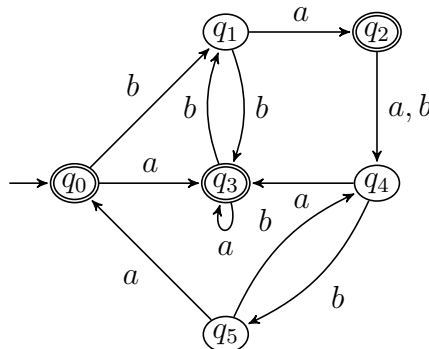
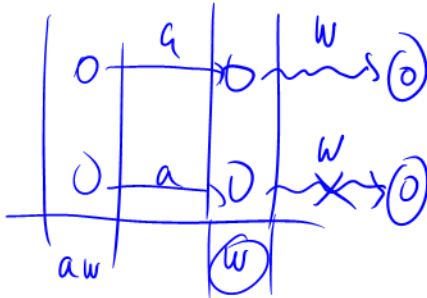
$$\text{or } b^* a^+ b^+ a^+ b^+ (b^+ a^+)^*$$

3 Minimisation of DFA

$$(a+b)^* = \varepsilon + (a+b) + (a+b)(a+b)^+ \dots$$

1.13

Exercise: Minimise the following DFA.

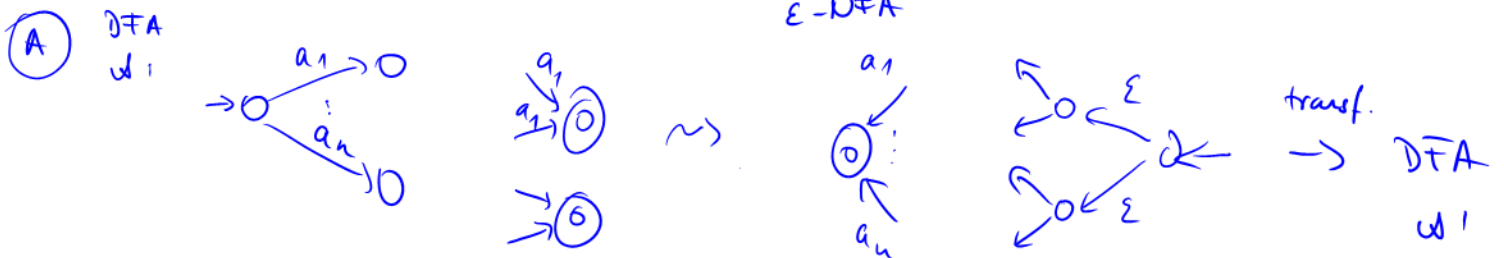


a ⁺	b ⁺	a ⁺	b ⁺
a ⁺	b ⁺	a ⁺	b ⁺
a ⁺	b ⁺	a ⁺	b ⁺
b ⁺	a ⁺	b ⁺	a ⁺
b ⁺	a ⁺	b ⁺	a ⁺
b ⁺	a ⁺	b ⁺	a ⁺

1.12

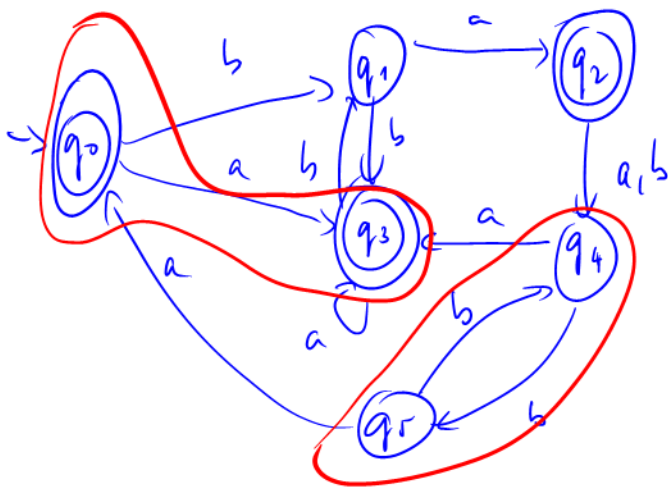
$$w = w_1 \dots w_n \quad w^k = w \dots w$$

- (A) $L \subseteq L \rightarrow \exists \text{ DFA } d. d(w) = L \Rightarrow \text{construct DFA } d'. d'(w) = L^k$
- (B) $L \subseteq L \rightarrow \exists \text{ reg. expr. } e. d(e) = L \rightarrow \text{construct reg. expr. } e'. d(e') = L^k$



$$e ::= \phi \mid \varepsilon \mid a \mid e + e \mid e \cdot e \mid e^* \mid (e)$$

$$\begin{aligned} e = \phi & : e' = \phi & e = e_1 + e_2 & : e' = e'_1 + e'_2 & e = (e_1) & : (e'_1) \\ e = \varepsilon & : e' = \varepsilon & e = e_1 \cdot e_2 & : e' = e'_2 \cdot e'_1 \\ e = a & : e' = a & e = e_1^* & : e' = (e'_1)^* \end{aligned}$$



	0	1	2	3	4	5
0	-	ϵ	a ✓		ϵ	ϵ
1	-	-	ϵ	ϵ	aa ✓	aa ✓
2	-	-	-	a ✓	ϵ	ϵ
3	-	-	-	-	ϵ	ϵ
4	-	-	-	-	-	-
5	-	-	-	✓	-	-

$$L_1 = L_2 \Leftrightarrow (L_1 \cap \bar{L}_2) \cup (L_2 \cap \bar{L}_1) = \emptyset$$



$$\begin{array}{l} q_0 \xrightarrow{a} q_3 \\ q_2 \xrightarrow{a} q_4 \end{array} \epsilon$$

$$\begin{array}{l} q_2 \xrightarrow{a} q_4 \\ q_3 \xrightarrow{a} q_3 \end{array} \epsilon$$

$$\begin{array}{l} q_1 \xrightarrow{a} q_2 \\ q_4 \xrightarrow{a} q_3 \end{array} a$$

$$\begin{array}{l} q_1 \xrightarrow{a} q_2 \\ q_5 \xrightarrow{a} q_0 \end{array} a$$

Minimal DFA:

