

Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Processes
Part A: Regular Languages
b-it Bonn; March 12–16, 2018

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Thanks to Thomas Noll for providing slides

https://ths.rwth-aachen.de/teaching/ws18/b-it-bridging-course/

Organisation

Schedule:

- lecture 9:00-10:30, 11:00-12:30 (Mon-Fri)

  10:00-11:30, 11:45-13:15?

- exercises 14:00-14:45, 15:15-16:00 (Mon-Fri)

  14:00-15:30?

Please ask questions!

Monday

9:00 – 10:30 11:00 – 12:30

13:15 – 14:45

Overview of Week 3

1. Regular Languages
   - Formal Languages
   - Finite Automata
   - Regular Expressions
   - Minimisation of Finite Automata
2. Context-Free Languages
   - Context-Free Grammars and Languages
   - Context-Free vs. Regular Languages
   - The Word Problem for Context-Free Languages
   - The Emptiness Problem for Context-Free Languages
   - Closure Properties of Context-Free Languages

Literature

J.E. Hopcroft, R. Motwani, J.D. Ullmann: ***Introduction to Automata Theory, Languages, and Computation***, 2nd ed., Addison-Wesley, 2001

A. Asteroth, C. Baier: ***Theoretische Informatik***, Pearson Studium, 2002 [in German]

http://www.jflap.org/
(software for experimenting with formal languages and automata)

Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

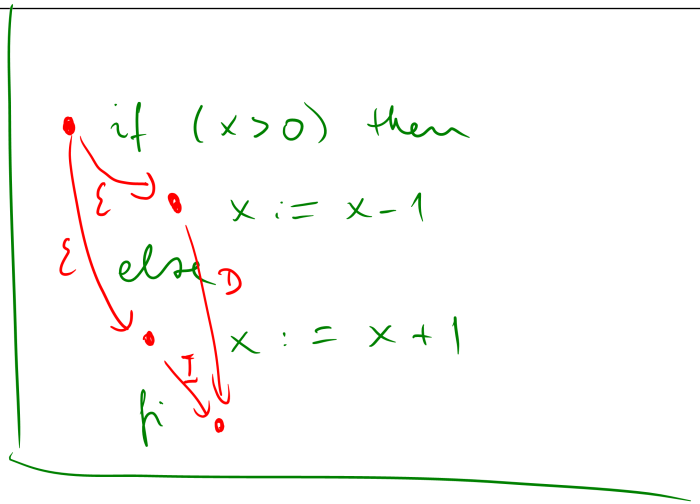
Outlook

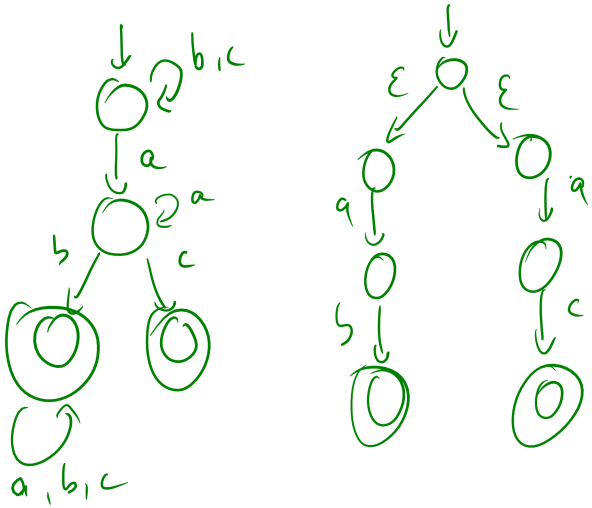
Words and Languages

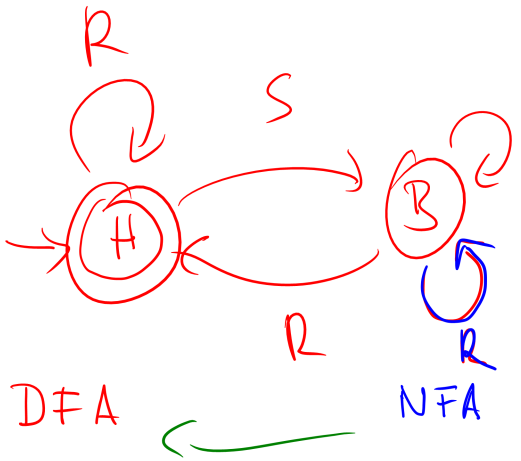
Computer systems transform data

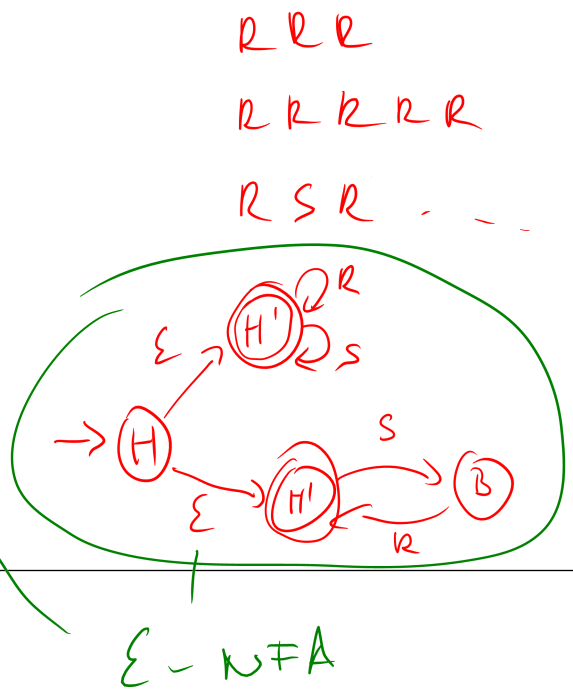
Data encoded as (binary) **words**

⇒ Data sets = sets of words = **formal languages**,
data transformations = **functions on words**









Words and Languages

Computer systems transform data

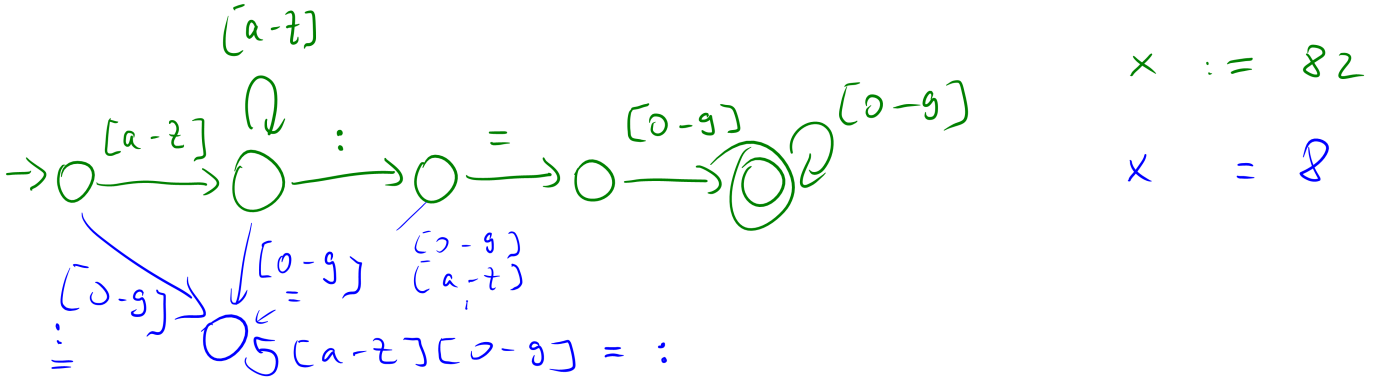
Data encoded as (binary) **words**

⇒ Data sets = sets of words = **formal languages**,
data transformations = **functions on words**

Example A.1

Java = {all valid Java programs}

Compiler : *Java* → *Bytecode*



Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An **alphabet** is a finite, non-empty set of symbols (“letters”).

$\Sigma, \Gamma, \ldots$ denote alphabets

$a, b, \ldots$ denote letters

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1. Boolean alphabet $\mathbb{B} := \{0, 1\}$

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2. Latin alphabet $\Sigma_{\text{latin}} := \{a, b, c, \ldots, z\}$

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3. Keyboard alphabet Σ_{key}

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2. Latin alphabet $\Sigma_{\text{latin}} := \{a, b, c, \ldots, z\}$
3. Keyboard alphabet Σ_{key}
4. Morse alphabet $\Sigma_{\text{morse}} := \{\cdot, -, \sqcup\}$

Words

Definition A.4

A **word** is a finite sequence of letters from a given alphabet Σ.

Σ^* is the set of all words over Σ.

$\Sigma = \{0, 1\}$

$\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$

$\Sigma^+ = \{\not{\varepsilon}, 0, 1, 00, 01, 10, 11, \ldots\}$

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The **concatenation** of two words $v = a_1 \dots a_m$ ($m \in \mathbb{N}$) and $w = b_1 \dots b_n$ ($n \in \mathbb{N}$) is the word

$$v \cdot w := a_1 \dots a_m b_1 \dots b_n$$

(often written as vw).

Thus: $w \cdot \varepsilon = \varepsilon \cdot w = w$.

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If $w = a_1 \dots a_n$ then $w^R := a_n \dots a_1$.

Formal Languages I

Definition A.5

A set of words $L \subseteq \Sigma^*$ is called a **(formal) language** over Σ.

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Example A.6

1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101

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1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101
2. over $\Sigma = \{I, V, X, L, C, D, M\}$: set of all valid roman numbers

Formal Languages I

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1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101
2. over $\Sigma = \{I, V, X, L, C, D, M\}$: set of all valid roman numbers
3. over Σ_{key}: set of all valid Java programs

Formal Languages II

Seen:

Basic notions: alphabets, words

Formal languages as sets of words

Formal Languages II

Seen:

Basic notions: alphabets, words

Formal languages as sets of words

Open:

Description of computations on words?

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Minimisation of DFA

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Ex. slide 16

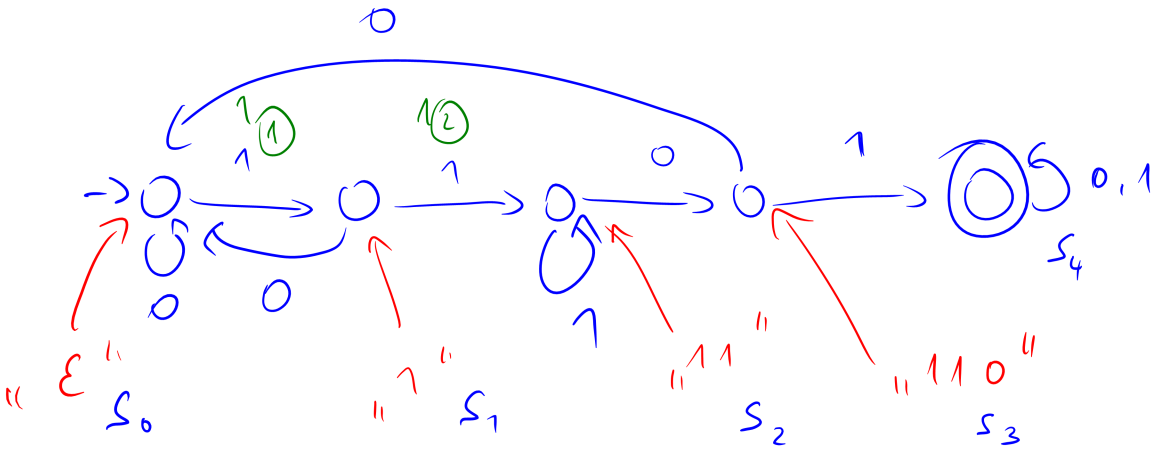
$\delta^*(s_0, 110) = s_3$

$\delta^*(s_1, 1100) = s_0$

Example: Pattern Matching

Example A.7 (Pattern 1101)

1. Read Boolean string bit-by-bit
2. Test whether it contains 1101
3. Idea: remember which (initial) part of 1101 has been recognised
4. Five prefixes: ε, 1, 11, 110, 1101
5. Diagram: on the board



1 1 0 1

1 1 1 0 1

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What we used:

finitely many (storage) states

an initial state

for every current state and every input symbol: a new state

a successful state

Deterministic Finite Automata I

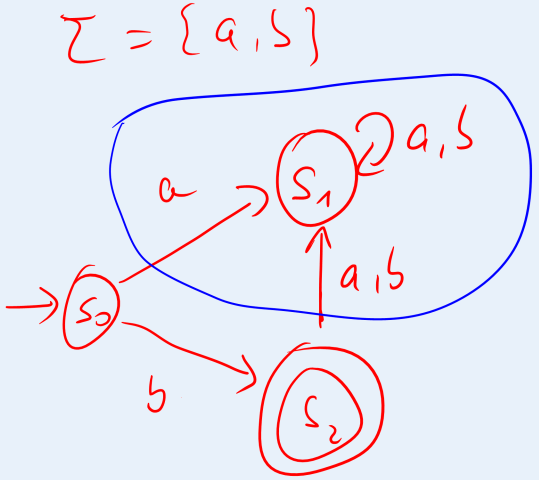
Definition A.8

A **deterministic finite automaton (DFA)** is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$$

where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\delta : Q \times \Sigma \to Q$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final** (or: **accepting**) **states**



$\delta(s_0, b) = s_2$

$s_0 \xrightarrow{b} s_2$

Deterministic Finite Automata II

Example A.9

Pattern matching (Example A.7):

$Q = \{q_0, \ldots, q_4\}$

$\Sigma = \mathbb{B} = \{0, 1\}$

$\delta : Q \times \Sigma \to Q$ on the board

$F = \{q_4\}$

Deterministic Finite Automata II

Example A.9

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$Q = \{q_0, \ldots, q_4\}$

$\Sigma = \mathbb{B} = \{0, 1\}$

$\delta : Q \times \Sigma \to Q$ on the board

$F = \{q_4\}$

Graphical Representation of DFA:

states $\implies$ nodes

$\delta(q, a) = q' \implies q \xrightarrow{a} q'$

initial state: incoming edge without source state

final state(s): double circle

Acceptance by DFA I

Definition A.10

Let $\langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA. The **extension** of $\delta : Q \times \Sigma \to Q$,

$$\delta^* : Q \times \Sigma^* \to Q,$$

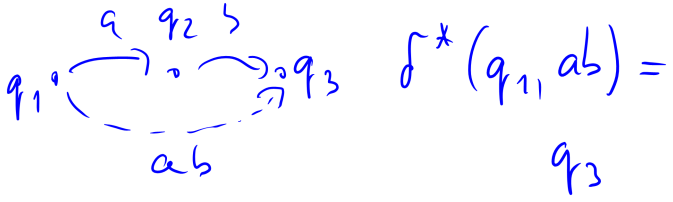
is defined by

$$\delta^*(q, w) := \text{state after reading } w \text{ starting from } q.$$

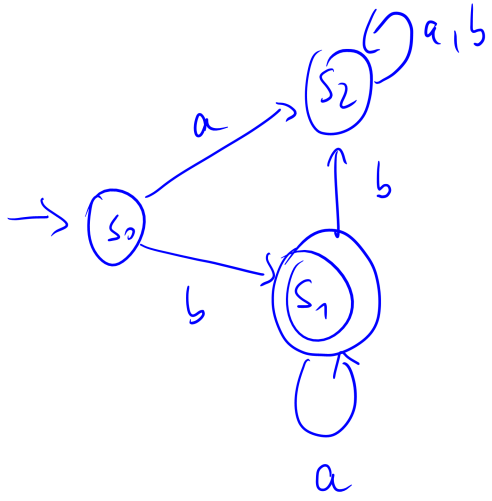
Formally:

$$\delta^*(q, w) := \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), v) & \text{if } w = av \end{cases}$$

Thus: if $w = a_1 \dots a_n$ and $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$, then $\delta^*(q, w) = q_n$



$\Sigma = \{a, b\}$



Example:

$\delta(s_0, b) = s_1$

$\delta^*(s_0, ba) = s_1$

$\delta^*(s_1, abab) = s_2$

$\delta : Q \times \Sigma \to Q$

$\delta^* : Q \times \Sigma^* \to Q$

$\delta^*(s, \varepsilon) = s$

$\delta^*(s, av) = \delta^*(\delta(s, a), v)$ $a \in \Sigma$, $v \in \Sigma^*$

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Example A.11

Pattern matching (Example A.9): on the board

Acceptance by DFA II

Definition A.12

$\mathfrak{A}$ **accepts** $w \in \Sigma^*$ if $\delta^*(q_0, w) \in F$.

The **language recognised (or: accepted) by** $\mathfrak{A}$ is

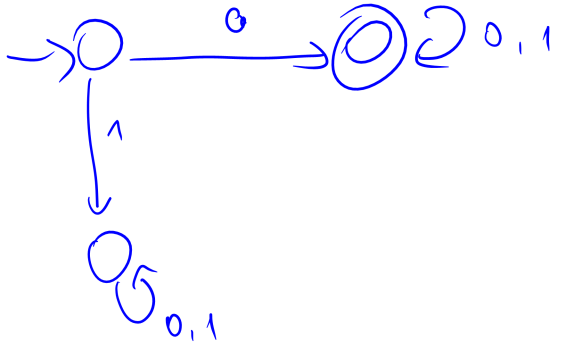
$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}.$$

A language $L \subseteq \Sigma^*$ is called **DFA-recognisable** if there exists some DFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.

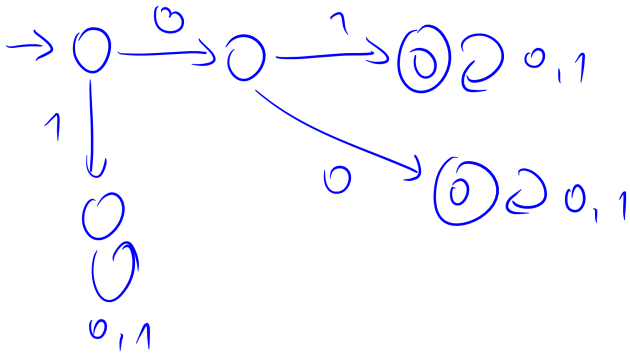
Two DFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called **equivalent** if

$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2).$$

$\Sigma = \{0, 1\}$



$|Q| = 3$



$|Q| = 5$

Acceptance by DFA III

Example A.13

1. The set of all bit strings containing 1101 is recognised by the automaton from Example A.9.

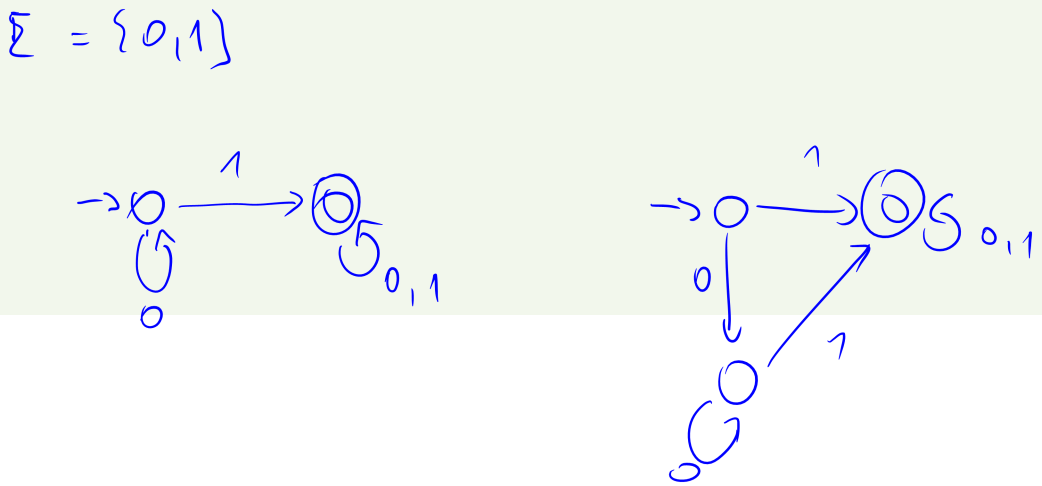
Acceptance by DFA III

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1. The set of all bit strings containing 1101 is recognised by the automaton from Example A.9.
2. Two (equivalent) automata recognising the language

$$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\} :$$

on the board



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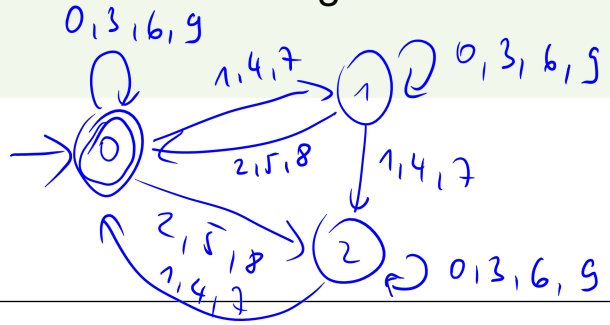
$$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\} :$$

   on the board
3. An automaton which recognises

$$\{w \in \{0,\ldots,9\}^* \mid \text{value of } w \text{ divisible by } 3\}$$

$\Sigma = \{0, \ldots, 9\}$

   Idea: test whether sum of digits is divisible by 3 – one state for each residue class (on the board)



13 $\quad 1+3=4$

12311 $\quad 1+2+3+4+1+1=$ 12

Deterministic Finite Automata

Seen:

Deterministic finite automata as a model of simple sequential computations

Recognisability of formal languages by automata

Deterministic Finite Automata

Seen:

Deterministic finite automata as a model of simple sequential computations

Recognisability of formal languages by automata

Open:

Composition and transformation of automata?

Which languages are recognisable, which are not (alternative characterisation)?

Language definition $\mapsto$ automaton and vice versa?

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Operations on Languages

Simplest case: Boolean operations (complement, intersection, union)

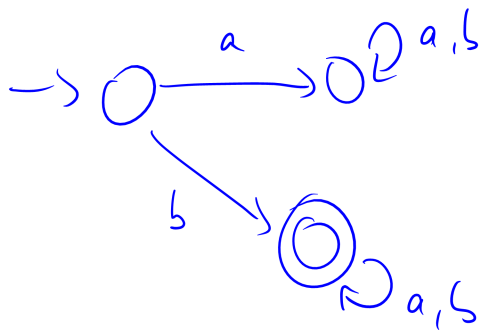
Question

Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two DFA with $L(\mathfrak{A}_1) = L_1$ and $L(\mathfrak{A}_2) = L_2$.
Can we construct automata which recognise

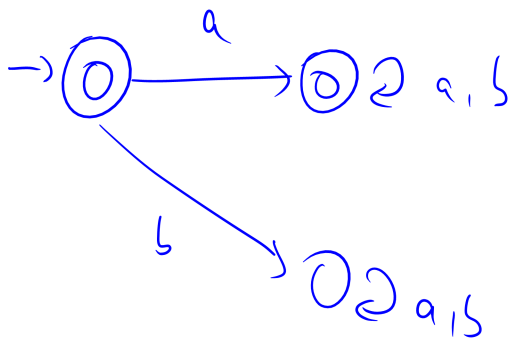
- $\overline{L_1}$ $(:= \Sigma^* \setminus L_1)$,
- $L_1 \cap L_2$, and
- $L_1 \cup L_2$?

$\Sigma = \{a, b\}$

$\mathcal{A}_1$:



$\mathcal{A}_2$:



$$\mathcal{L}(\mathcal{A}_1) = \{ bv \mid v \in \Sigma^* \} \qquad \mathcal{L}(\mathcal{A}_2) = \{ \varepsilon \} \cup \{ av \mid v \in \Sigma^* \}$$

$$\overline{\mathcal{L}(\mathcal{A}_1)} = \mathcal{L}(\mathcal{A}_2) = \Sigma^* \setminus \mathcal{L}(\mathcal{A}_1)$$

Language Complement

Theorem A.14

If $L \subseteq \Sigma^*$ is DFA-recognisable, then so is $\overline{L}$.

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Proof.

Let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA such that $L(\mathfrak{A}) = L$. Then:

$$w \in \overline{L} \iff w \notin L \iff \delta^*(q_0, w) \notin F \iff \delta^*(q_0, w) \in Q \setminus F.$$

Thus, $\overline{L}$ is recognised by the DFA $\langle Q, \Sigma, \delta, q_0, Q \setminus F \rangle$. □

Language Complement

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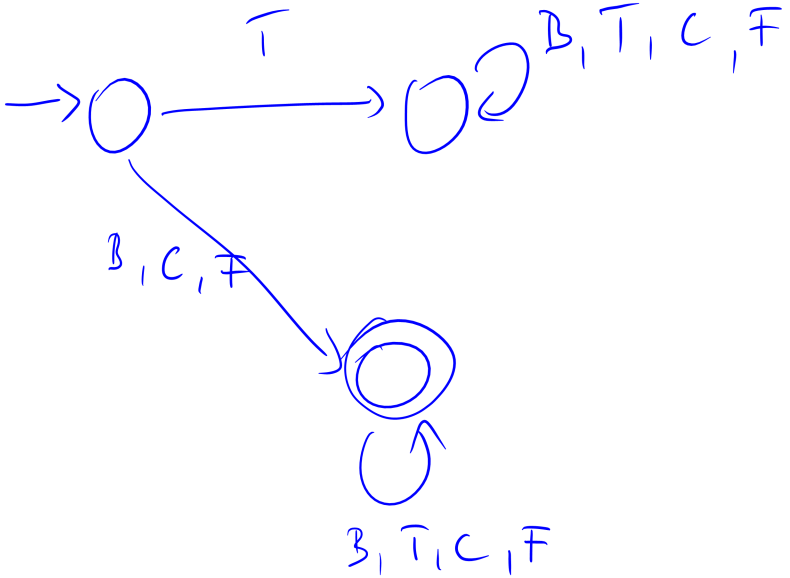
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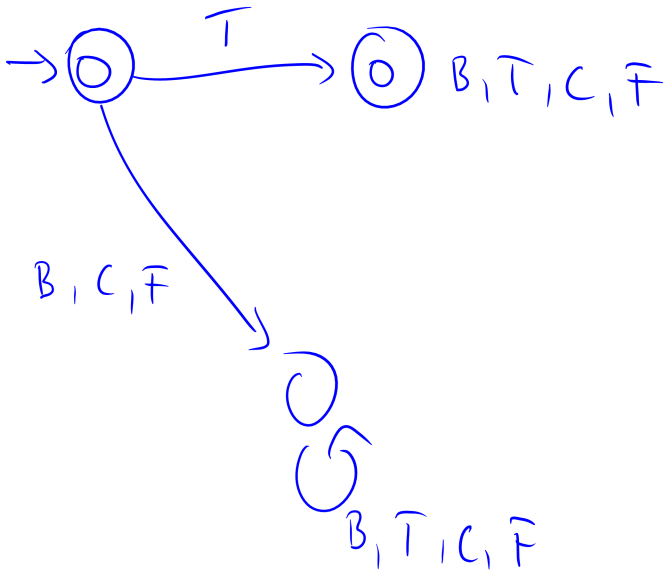
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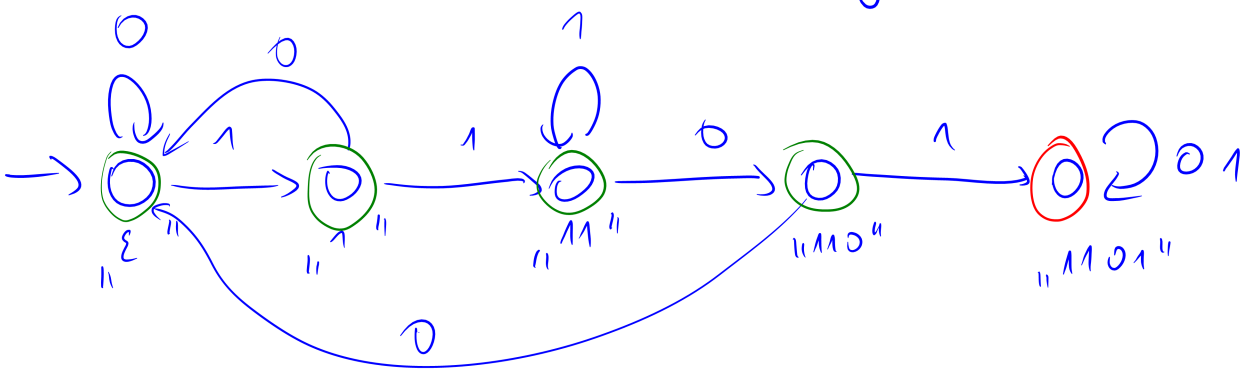
Example A.15

on the board


T
B, T, C, F
B, C, F
B, T, C, F


T
B, T, C, F
B, C, F
B, T, C, F

$\Sigma = \{0, 1\}$ contain string 1101



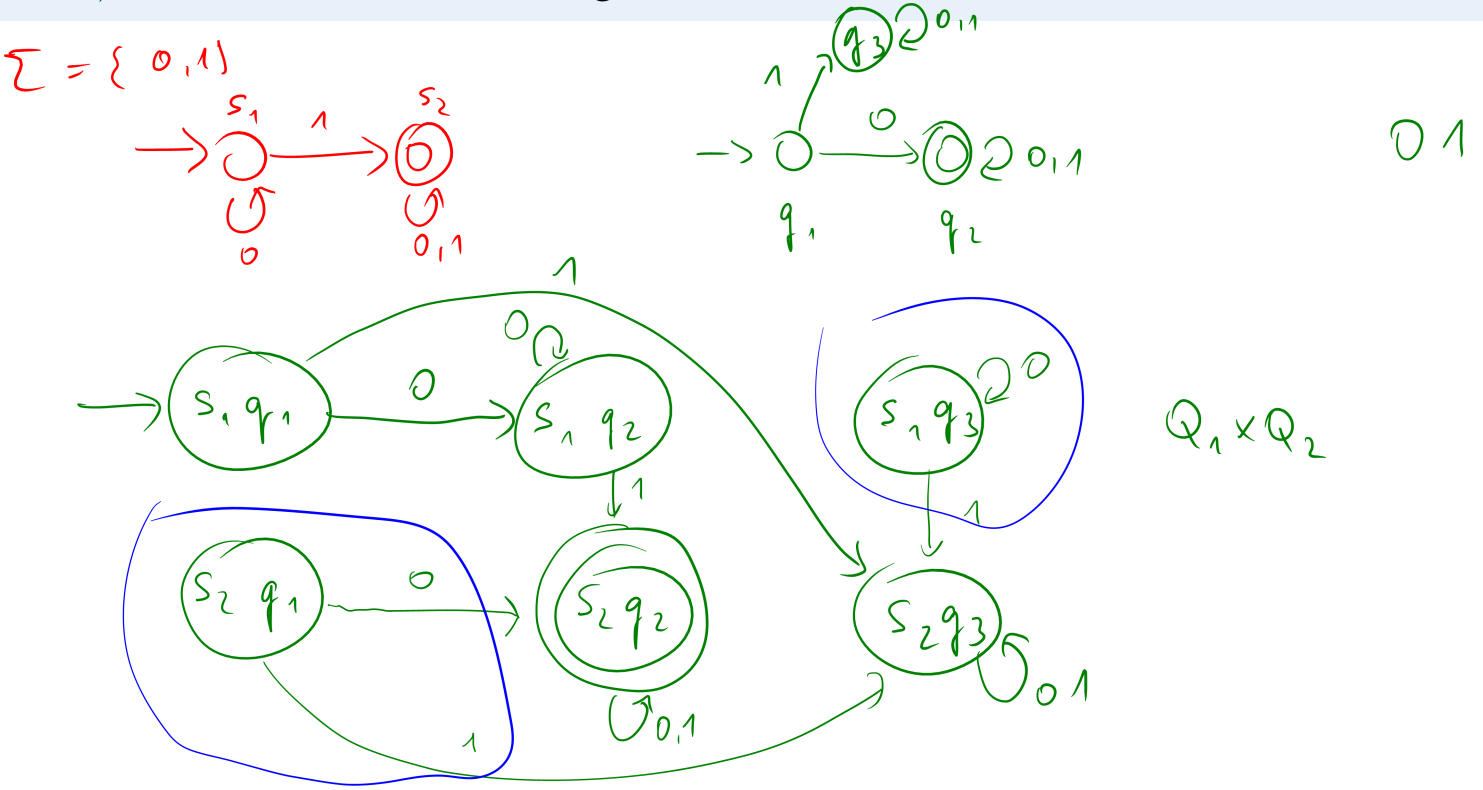
$$\mathcal{L}(\mathcal{A}) = \{ v\,1101\,w \mid v, w \in \Sigma^* \}$$

$$\mathcal{L}(\mathcal{A}) = \{ \varepsilon, 0\ldots0, 0\ldots01\,0\ldots0, \ldots \} = \overline{\mathcal{L}(\mathcal{A})}$$

Language Intersection I

Theorem A.16

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cap L_2$.



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Proof.

Let $\mathfrak{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathfrak{A}_i) = L_i$ ($i = 1, 2$). The new automaton $\mathfrak{A}$ has to accept w iff $\mathfrak{A}_1$ **and** $\mathfrak{A}_2$ accept w

Idea: let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ run **in parallel**

- use pairs of states $(q_1, q_2) \in Q_1 \times Q_2$
- start with both components in initial state
- a transition updates both components independently
- for acceptance **both** components need to be in a final state

□

Language Intersection II

Proof (continued).

Formally: let the **product automaton**

$$\mathfrak{A} := \langle Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), F_1 \times F_2 \rangle$$

be defined by

$$\delta((q_1, q_2), a) := (\delta_1(q_1, a), \delta_2(q_2, a)) \text{ for every } a \in \Sigma.$$

Language Intersection II

Proof (continued).

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This definition yields (for every $w \in \Sigma^*$):

$$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \qquad (*)$$

Language Intersection II

Proof (continued).

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Thus: $\mathfrak{A}$ accepts w

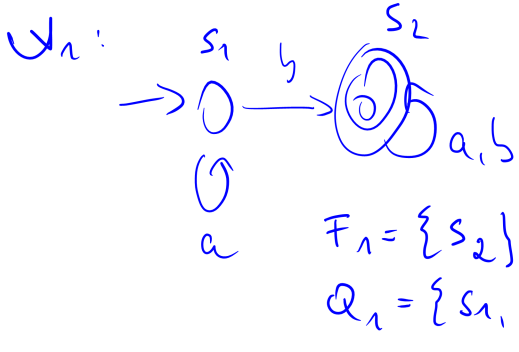
$$\begin{aligned} &\iff \delta^*((q_0^1, q_0^2), w) \in F_1 \times F_2 \\ &\overset{(*)}{\iff} (\delta_1^*(q_0^1, w), \delta_2^*(q_0^2, w)) \in F_1 \times F_2 \\ &\iff \delta_1^*(q_0^1, w) \in F_1 \text{ and } \delta_2^*(q_0^2, w) \in F_2 \\ &\iff \mathfrak{A}_1 \text{ accepts } w \text{ and } \mathfrak{A}_2 \text{ accepts } w \end{aligned}$$

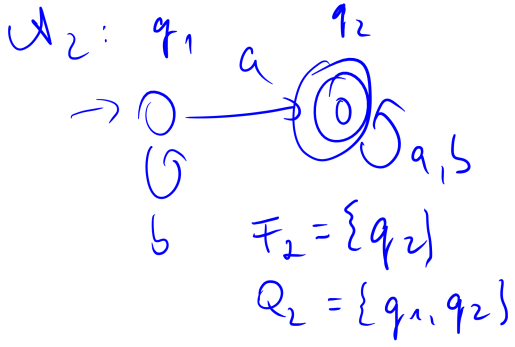
□

Example A.17

on the board

$\Sigma = \{a, b\}$



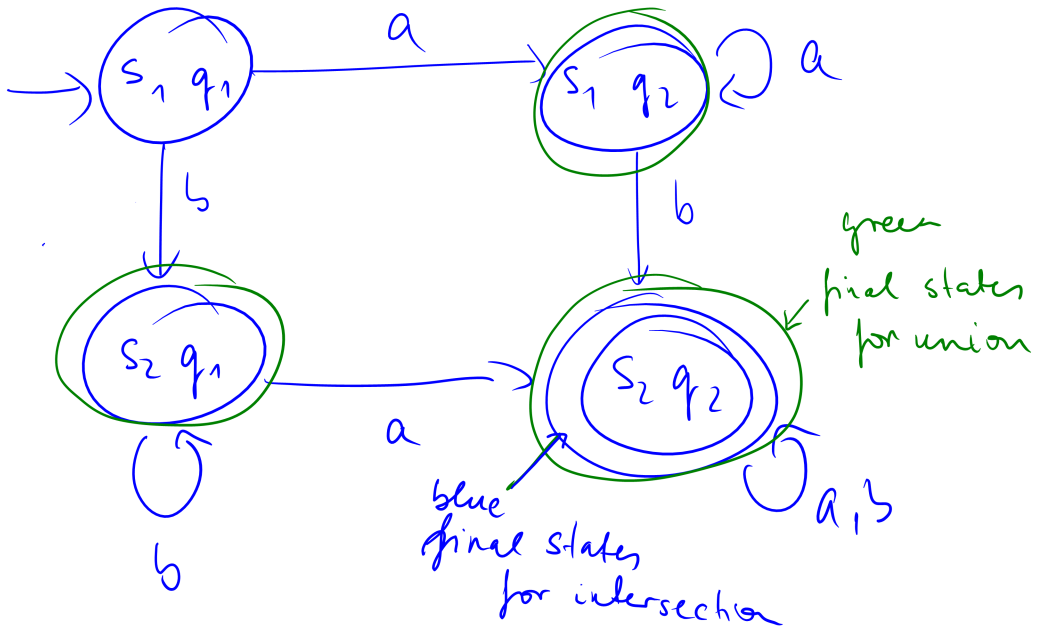


a a b b ✓

a a a x

$F = F_1 \times F_2 \Rightarrow$

$L = L_1 \cap L_2$



$F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$



Language Union

Theorem A.18 0

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cup L_2$.

Language Union

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Idea: reuse product construction

Construct $\mathfrak{A}$ as before but choose as final states those pairs $(q_1, q_2) \in Q_1 \times Q_2$ with $q_1 \in F_1$ **or** $q_2 \in F_2$. Thus the set of final states is given by

$$F := (F_1 \times Q_2) \cup (Q_1 \times F_2).$$

$F_1 \times F_2$

□

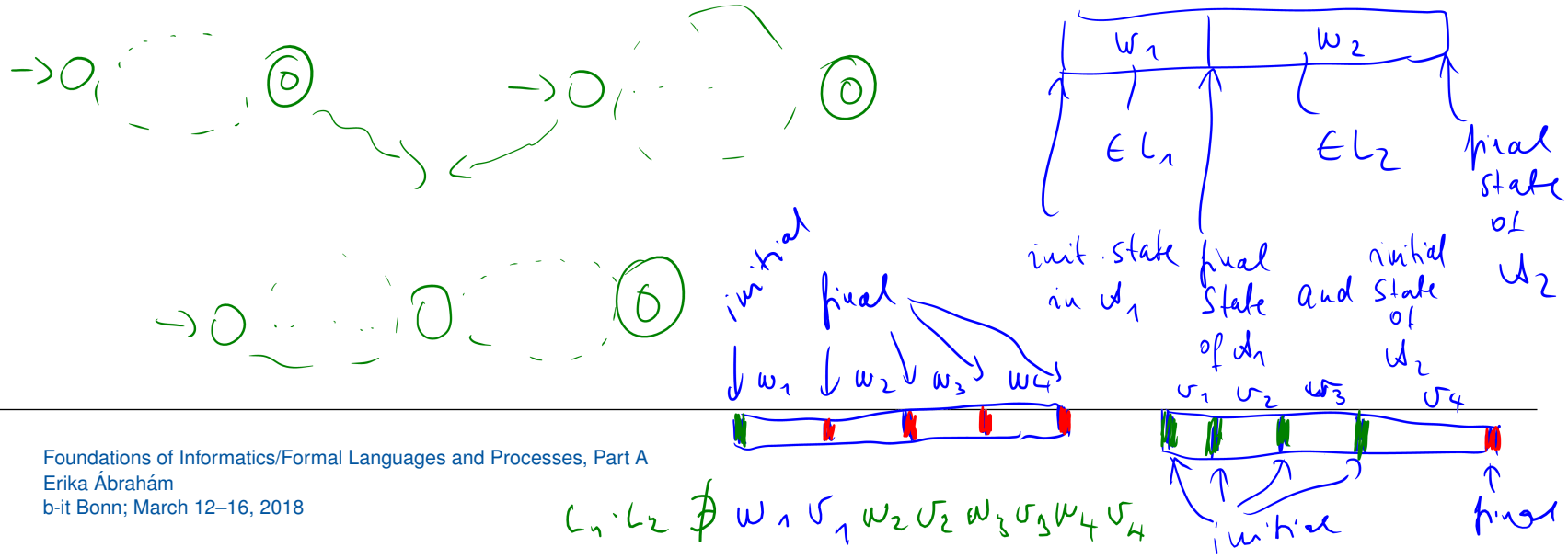
Language Concatenation

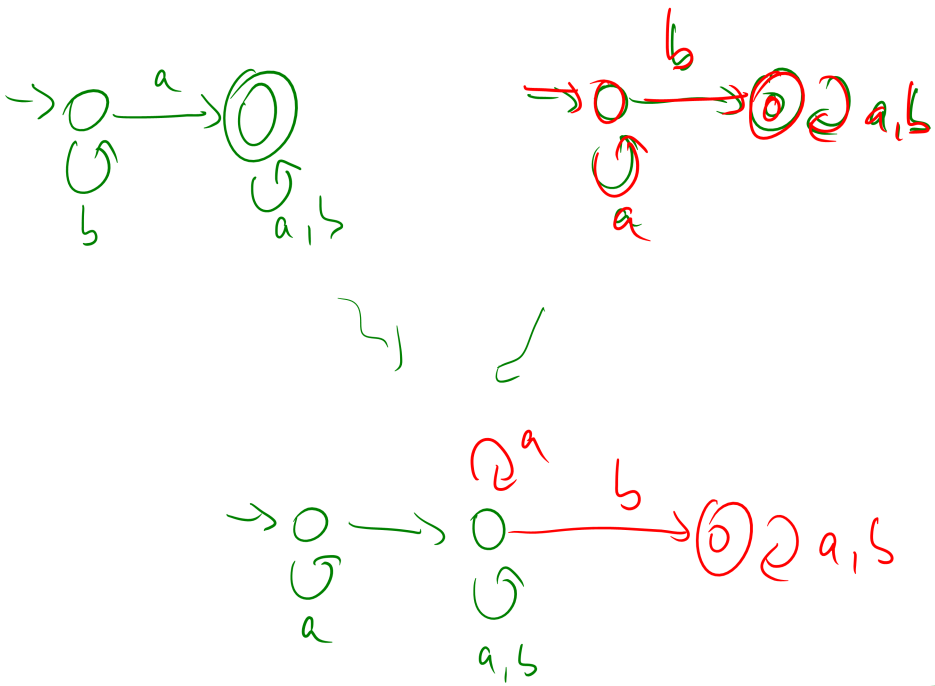
Definition A.19

The **concatenation** of two languages $L_1, L_2 \subseteq \Sigma^*$ is given by

$$L_1 \cdot L_2 := \{v \cdot w \in \Sigma^* \mid v \in L_1, w \in L_2\}.$$

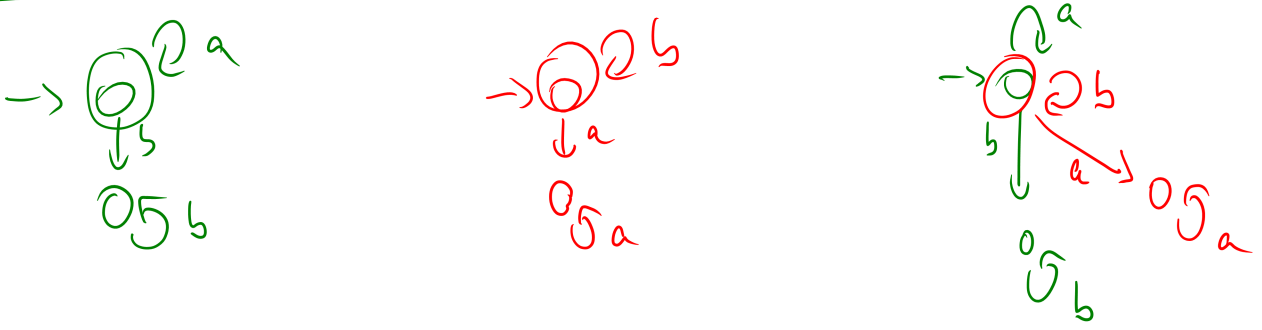
Abbreviations: $w \cdot L := \{w\} \cdot L$, $L \cdot w := L \cdot \{w\}$





Problems

1. non-determinism
   ↓
   NFA



2. semantically "buggy" due to loops on final states

Language Concatenation

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Example A.20

1. If $L_1 = \{101, 1\}$ and $L_2 = \{011, 1\}$, then
$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$

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Example A.20

1. If $L_1 = \{101, 1\}$ and $L_2 = \{011, 1\}$, then
$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$
2. If $L_1 = 00 \cdot \mathbb{B}^*$ and $L_2 = 11 \cdot \mathbb{B}^*$, then
$$L_1 \cdot L_2 = \{w \in \mathbb{B}^* \mid w \text{ has prefix } 00 \text{ and contains } 11\}.$$

DFA-Recognisability of Concatenation

Conjecture

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cdot L_2$.

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Idea: choose $Q := Q_1 \cup Q_2$ where each $q \in F_1$ is identified with q_0^2

But: on the board □

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But: on the board □

Conclusion

Required: automata model where the successor state (for a given state and input symbol) is **not unique**

Language Iteration

Definition A.21

The *n*th **power** of a language $L \subseteq \Sigma^*$ is the n-fold concatenation of L with itself ($n \in \mathbb{N}$):

$$L^n := \underbrace{L \cdot \ldots \cdot L}_{n \text{ times}} = \{w_1 \ldots w_n \mid \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$

Inductively: $L^0 := \{\varepsilon\}$, $L^{n+1} := L^n \cdot L$

The **iteration** (or: **Kleene star**) of L is

$$L^* := \bigcup_{n \in \mathbb{N}} L^n = \{w_1 \ldots w_n \mid n \in \mathbb{N}, \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$

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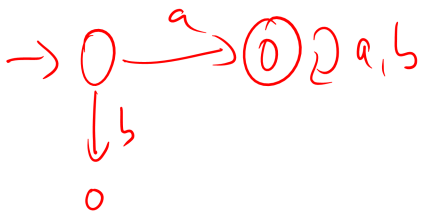
Remarks:

we always have $\varepsilon \in L^*$ (since $L^0 \subseteq L^*$ and $L^0 = \{\varepsilon\}$)

$w \in L^*$ iff $w = \varepsilon$ or if w can be decomposed into $n \geq 1$ subwords $v_1, \ldots, v_n$ (i.e., $w = v_1 \cdot \ldots \cdot v_n$) such that $v_i \in L$ for every $1 \leq i \leq n$

again we would suspect that the iteration of a DFA-recognisable language is DFA-recognisable, but there is no simple (deterministic) construction

A :

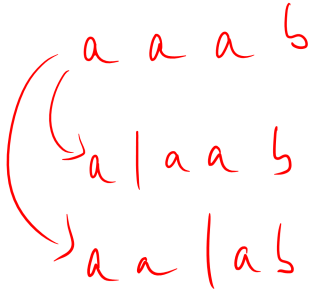


$\mathcal{L}(A) = L$ $\quad \Sigma = \{a, b\}$

$L^0 = \{\varepsilon\}$

$L^1 = \{a v \mid v \in \Sigma^*\}$

$L^2 = \{a v_1 \mid a v_2 \mid v_1, v_2 \in \Sigma^*\}$



Operations on Languages and Automata

Seen:

Operations on languages:

- complement
- intersection
- union
- concatenation
- iteration

DFA constructions for:

- complement
- intersection
- union

Operations on Languages and Automata

Seen:

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DFA constructions for:

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Open:

Automata model for (direct implementation of) concatenation and iteration?

Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

Nondeterministic Finite Automata I

Idea:

for a given state and a given input symbol, **several transitions** (or none at all) are possible

an input word generally induces **several state sequences** (“runs”)

the word is accepted if **at least one** accepting run exists

Nondeterministic Finite Automata I

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for a given state and a given input symbol, **several transitions** (or none at all) are possible

an input word generally induces **several state sequences** (“runs”)

the word is accepted if **at least one** accepting run exists

Advantages:

simplifies representation of languages

– example: $\mathbb{B}^* \cdot 1101 \cdot \mathbb{B}^*$ (on the board)

yields direct constructions for concatenation and iteration of languages

more adequate modelling of systems with nondeterministic behaviour

– communication protocols, multi-agent systems, ...

Nondeterministic Finite Automata II

Definition A.22

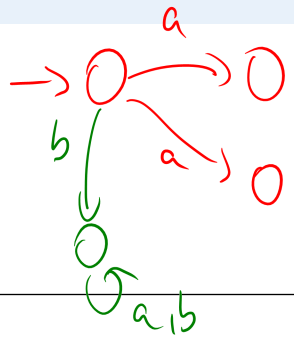
A **nondeterministic finite automaton (NFA)** is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\Delta \subseteq Q \times \Sigma \times Q$ is the **transition relation**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

before for DFA:
$\delta : Q \times \Sigma \to Q$



Nondeterministic Finite Automata II

Definition A.22

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Remarks:

- $(q, a, q') \in \Delta$ usually written as $q \xrightarrow{a} q'$
- every DFA can be considered as an NFA ($(q, a, q') \in \Delta \iff \delta(q, a) = q'$)

Acceptance by NFA

Definition A.23

Let $w = a_1 \dots a_n \in \Sigma^*$.

A w-labelled $\mathfrak{A}$-**run** from q_1 to q_2 is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots p_{n-1} \xrightarrow{a_n} p_n$$

such that $p_0 = q_1$, $p_n = q_2$, and $(p_{i-1}, a_i, p_i) \in \Delta$ for every $1 \leq i \leq n$ (we also write: $q_1 \xrightarrow{w} q_2$).

$\mathfrak{A}$ **accepts** w if there is a w-labelled $\mathfrak{A}$-run from q_0 to some $q \in F$

The **language recognised by** $\mathfrak{A}$ is

$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}.$$

A language $L \subseteq \Sigma^*$ is called **NFA-recognisable** if there exists a NFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.

Two NFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called **equivalent** if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Acceptance Test for NFA

Algorithm A.24 (Acceptance Test for NFA)

Input: *NFA* $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$, $w \in \Sigma^*$

Question: $w \in L(\mathfrak{A})$?

Procedure: *Computation of the* ***reachability set***

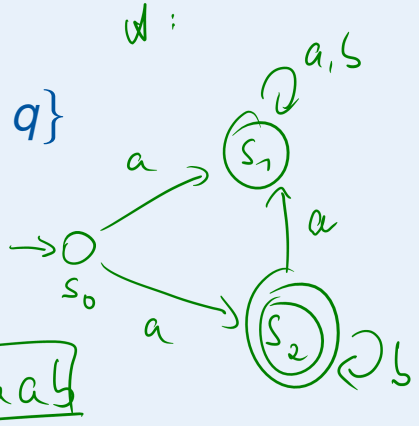
$$R_{\mathfrak{A}}(w) := \{q \in Q \mid q_0 \xrightarrow{w} q\}$$

Iterative procedure for $w = a_1 \ldots a_n$*:*

1. *let* $R_{\mathfrak{A}}(\varepsilon) := \{q_0\}$
2. *for* $i := 1, \ldots, n$*: let*

$$R_{\mathfrak{A}}(a_1 \ldots a_i) := \{q \in Q \mid \exists p \in R_{\mathfrak{A}}(a_1 \ldots a_{i-1}) : p \xrightarrow{a_i} q\}$$

Output: *"yes" if* $R_{\mathfrak{A}}(w) \cap F \neq \emptyset$*, otherwise "no"*



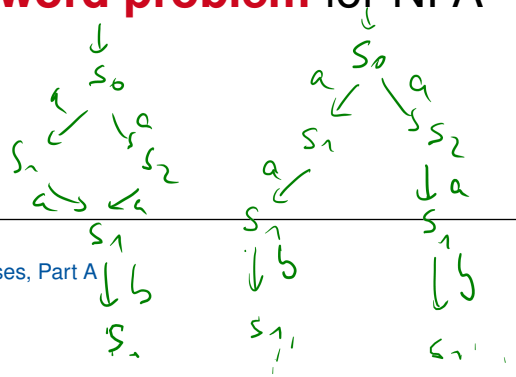
$R_{\mathfrak{A}}(\varepsilon) = \{s_0\}$

$R_{\mathfrak{A}}(a) = \{s_1, s_2\}$

$R_{\mathfrak{A}}(aa) = \{s_1\}$

$R_{\mathfrak{A}}(aab) = \{s_1\}$

Remark: this algorithm solves the **word problem** for NFA



Acceptance Test for NFA

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Output: *"yes" if* $R_{\mathfrak{A}}(w) \cap F \neq \emptyset$*, otherwise "no"*

Remark: this algorithm solves the **word problem** for NFA

Example A.25

on the board

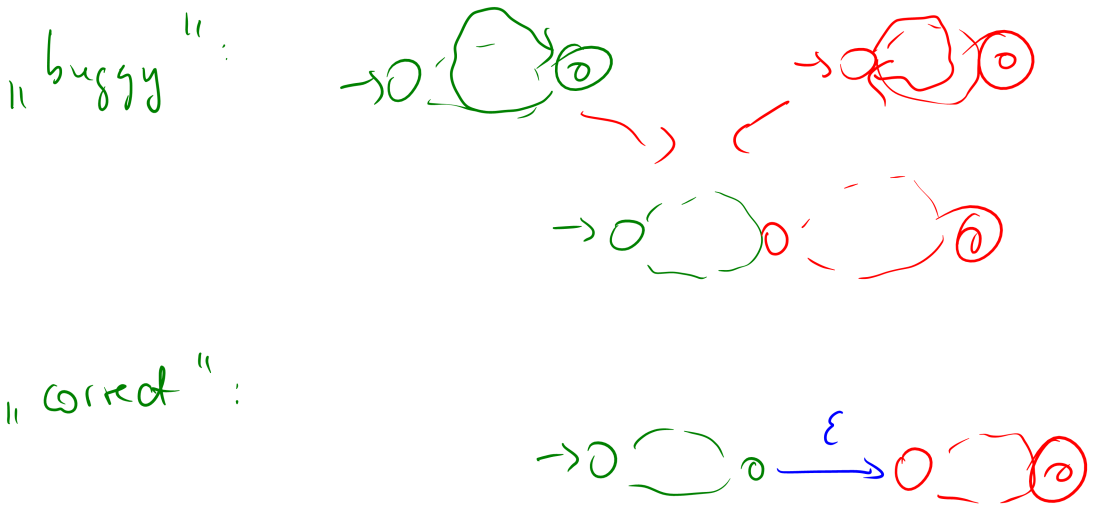
NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

Solution: admit **empty word ε as transition label**



ε-NFA

Definition A.26

A **nondeterministic finite automaton with ε-transitions (ε-NFA)** is of the form $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\Delta \subseteq Q \times \Sigma_\varepsilon \times Q$ is the **transition relation** where $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$
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Remarks:

- every NFA is an ε-NFA
- definitions of runs and acceptance: in analogy to NFA

ε-NFA

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Remarks:

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Example A.27

on the board

a a a a

{ or a ... a or b ... b

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognisable, then so is $L_1 \cdot L_2$.

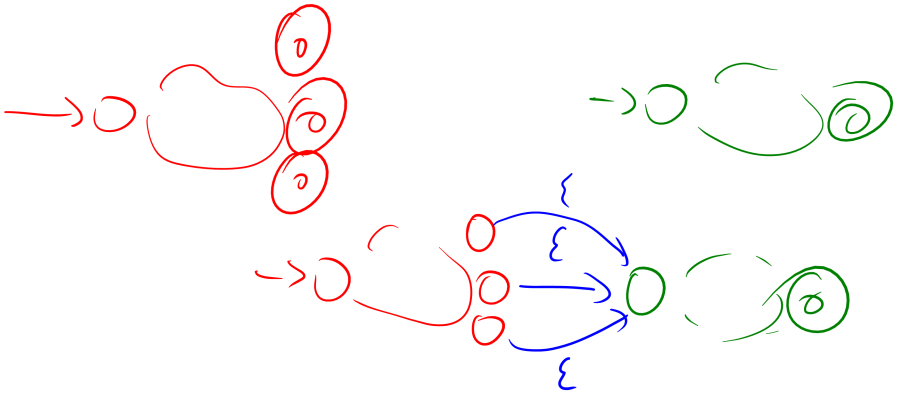
Concatenation and Iteration via ε-NFA

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Proof (idea).

on the board □



Concatenation and Iteration via ε-NFA

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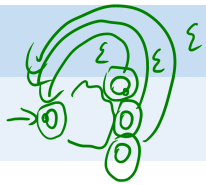
on the board □

Theorem A.29

If $L \subseteq \Sigma^*$ is ε-NFA-recognisable, then so is L^*.

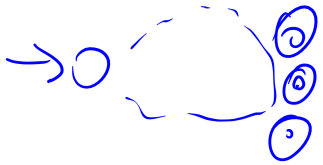
$L \cdot L \cdot L \cdot L \cdots$ arbitrary many ???

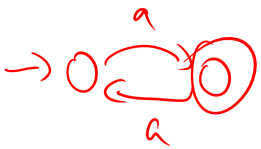
Proof.



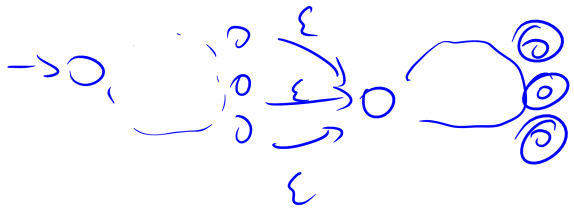
see Theorem A.47 □

A $\mathcal{L}(A) = L$

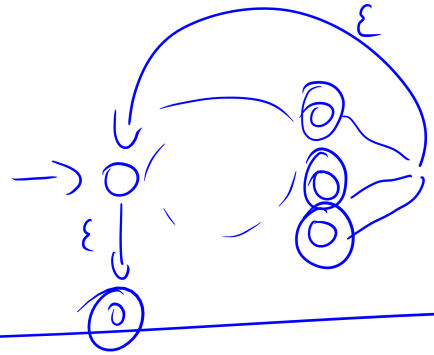




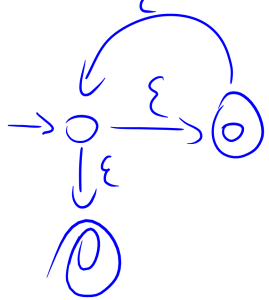
L^2:



L^*:





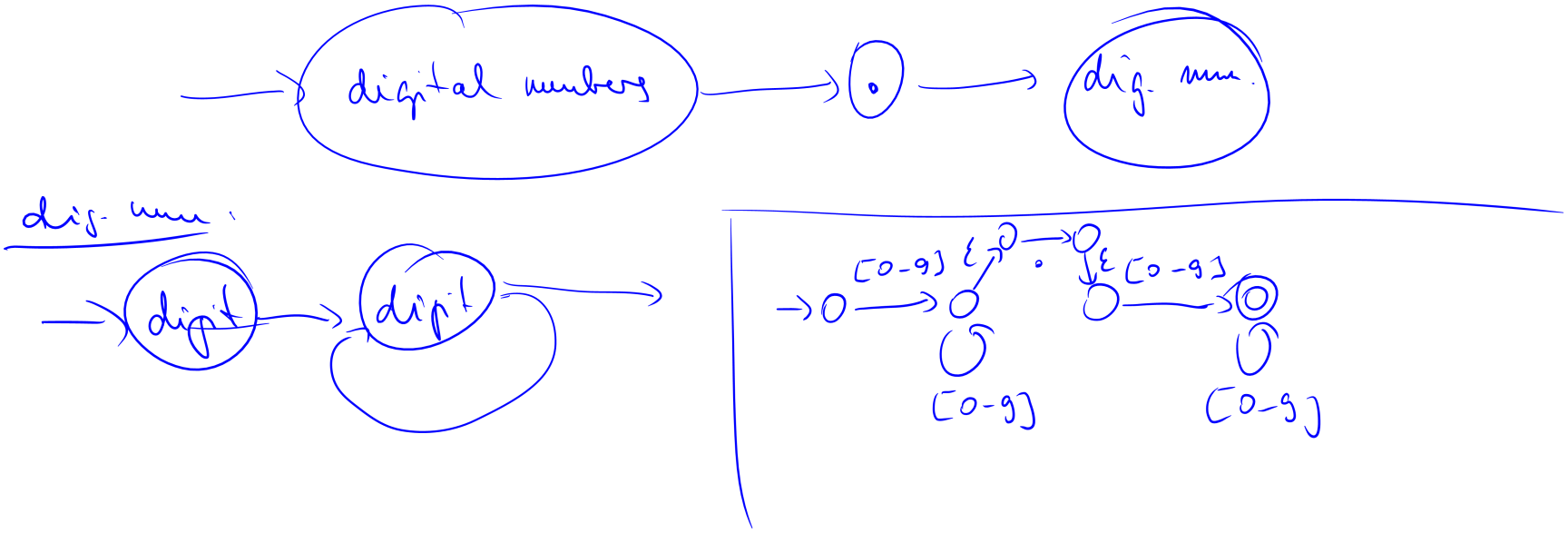


Syntax Diagrams as ε-NFA

Syntax diagrams (without recursive calls) can be interpreted as ε-NFA

Example A.30

decimal numbers (on the board)

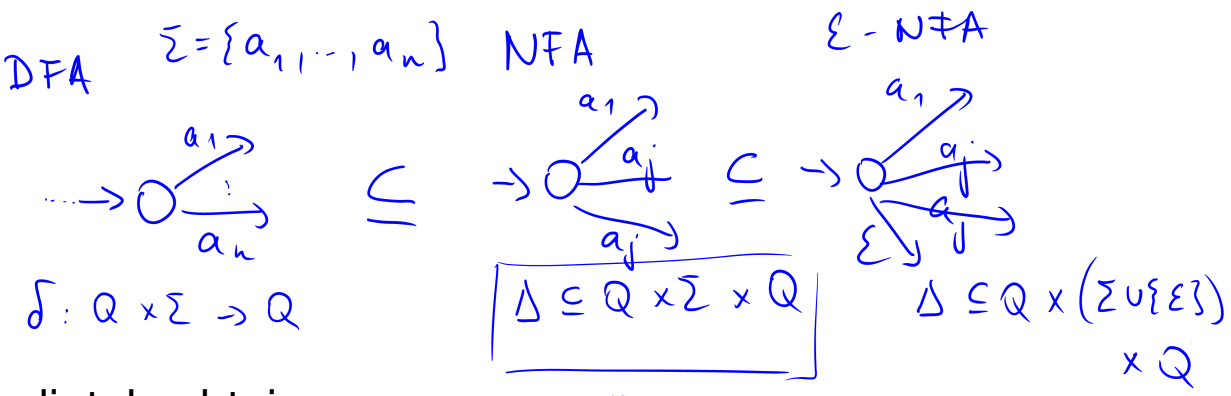


Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε-NFA (Definition A.26)

Types of Finite Automata

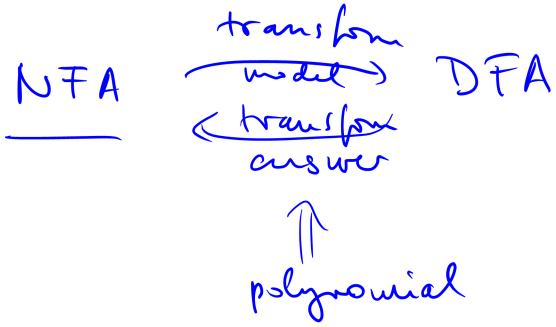
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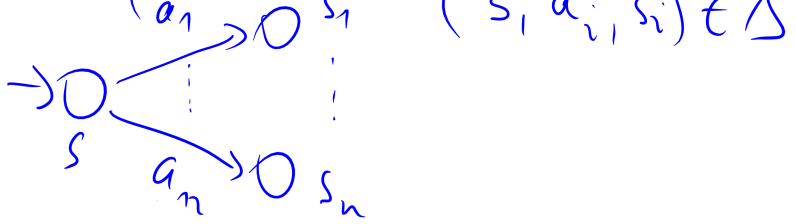


From the definitions we immediately obtain:

Corollary A.31

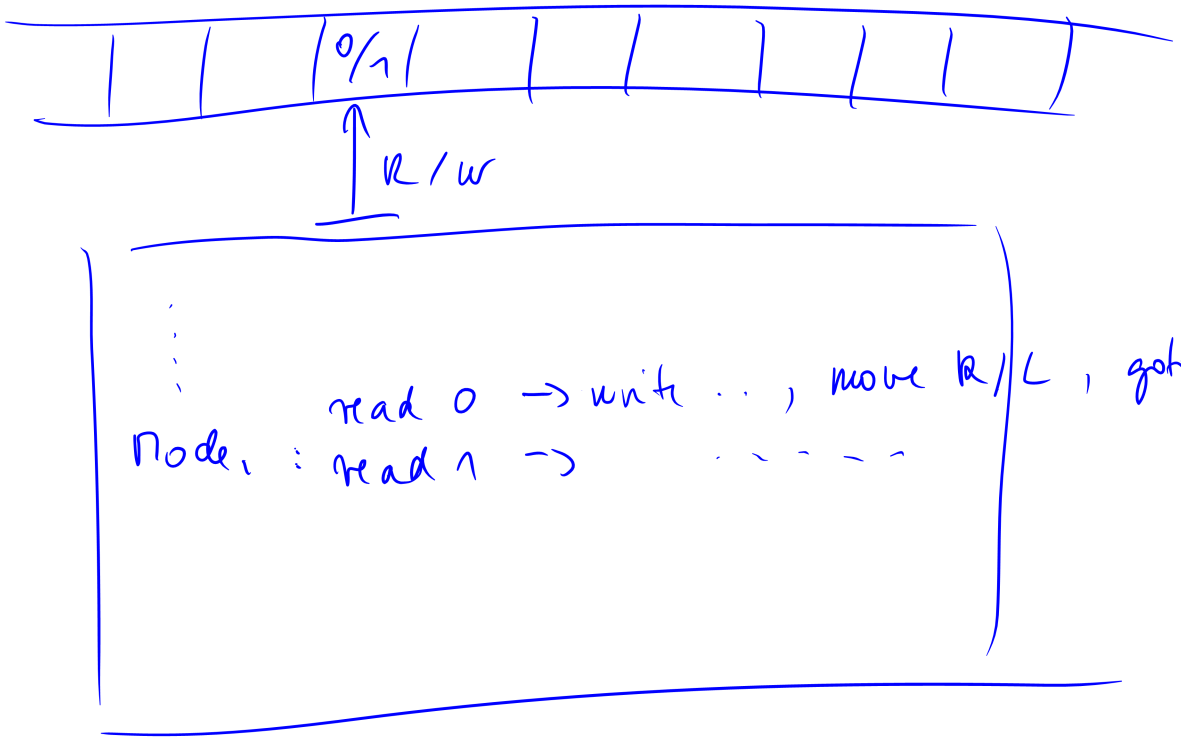
1. *Every DFA-recognisable language is NFA-recognisable.*
2. *Every NFA-recognisable language is ε-NFA-recognisable.*

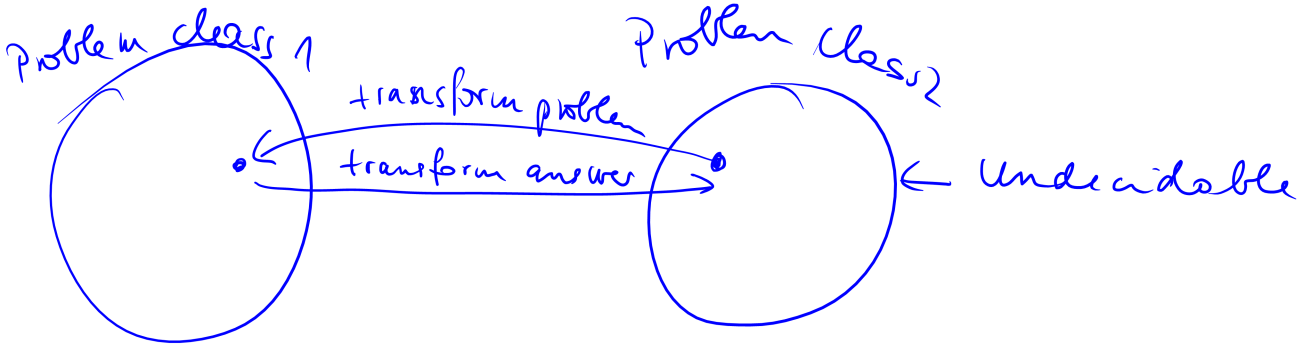




Turing - Maschine

$\Sigma = \{0, 1\}$





Types of Finite Automata

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2. NFA (Definition A.22)
3. ε-NFA (Definition A.26)

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Corollary A.31

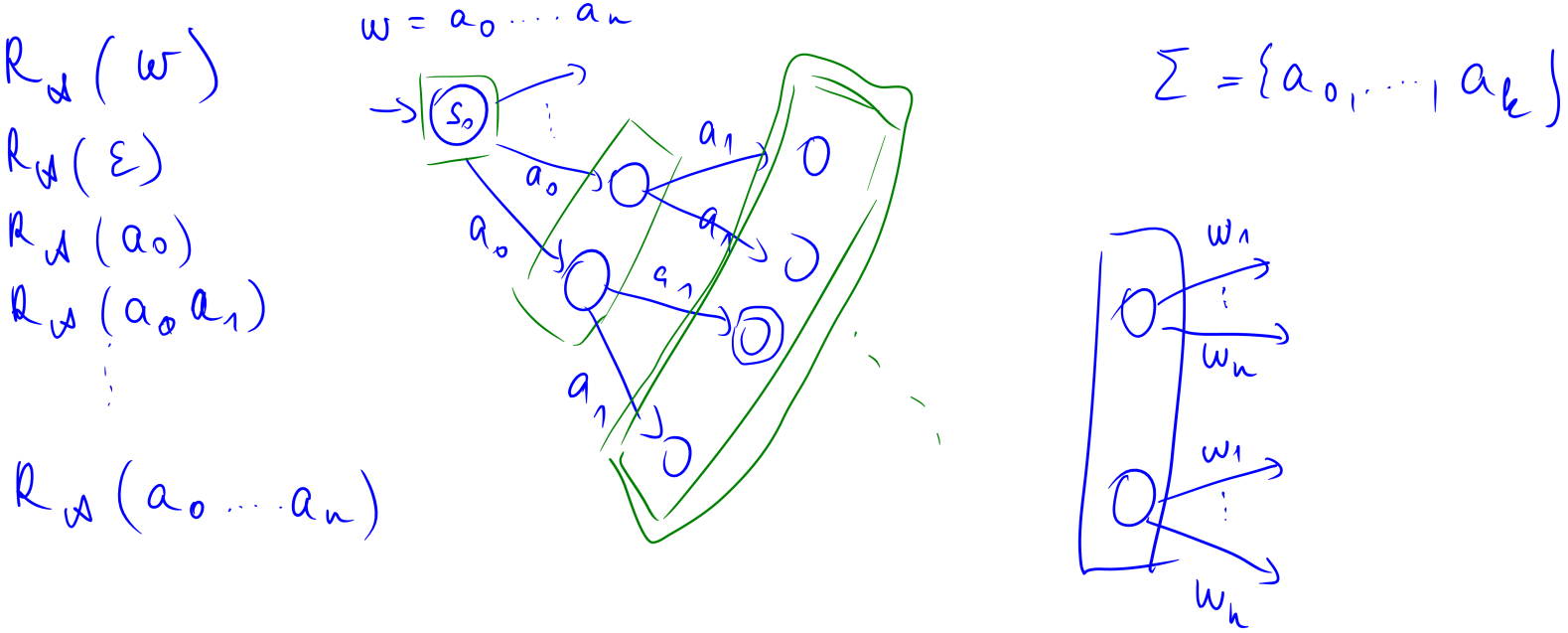
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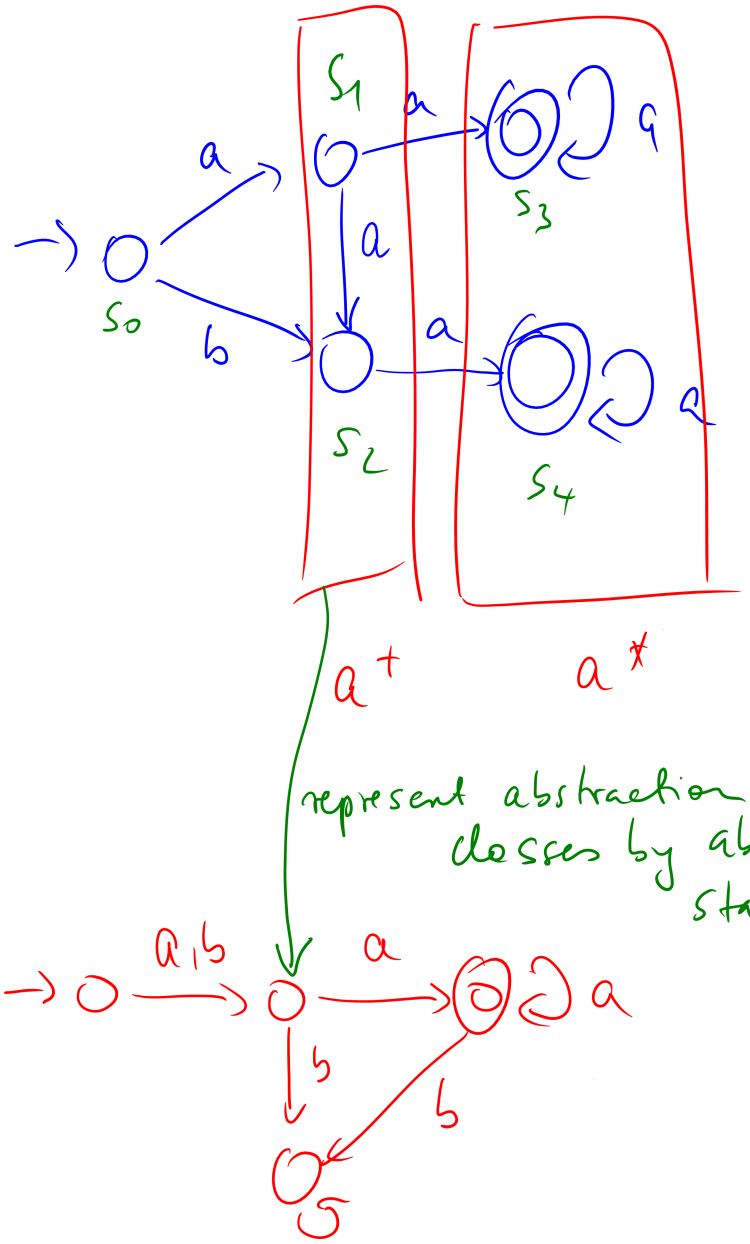
Goal: establish reverse inclusions

From NFA to DFA I

Theorem A.32

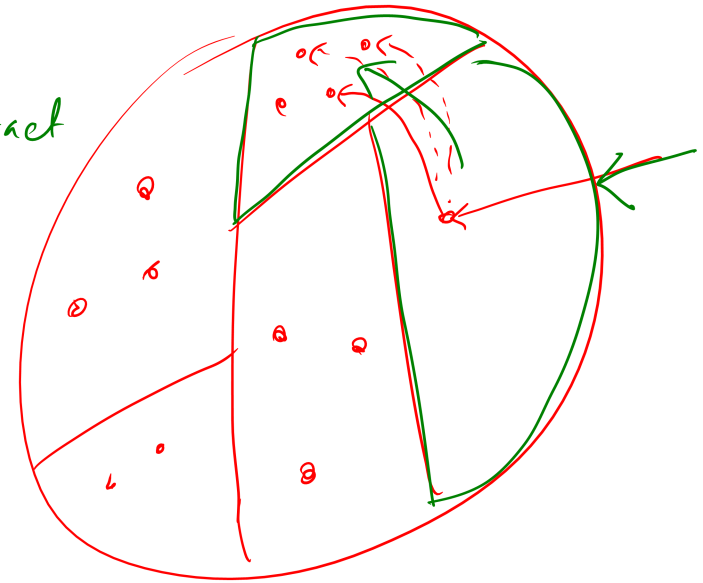
Every NFA can be transformed into an equivalent DFA.





$$R_{\mathcal{A}}(s_0) = \{ a\,a\,a^* \}$$

$$R_{\mathcal{A}}(s_1) = \{ a\,a^* \}$$



From NFA to DFA I

Theorem A.32

Every NFA can be transformed into an equivalent DFA.

Proof.

Idea: let the DFA operate on **sets of states** ("**powerset construction**")

Initial state of DFA := {initial state of NFA}

$P \xrightarrow{a} P'$ in DFA iff there exist $q \in P, q' \in P'$ such that $q \xrightarrow{a} q'$ in NFA

P final state in DFA iff it contains some final state of NFA

□

From NFA to DFA II

Proof (continued).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ a NFA. **Powerset construction** of $\mathfrak{A}' = \langle Q', \Sigma, \delta', q_0', F' \rangle$:

- $Q' := 2^Q := \{P \mid P \subseteq Q\}$
- $\delta' : Q' \times \Sigma \to Q'$ with $q \in \delta'(P, a) \iff$ there exists $p \in P$ such that $(p, a, q) \in \Delta$
- $q_0' := \{q_0\}$
- $F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$

This yields

$$q_0 \xrightarrow{w} q \text{ in } \mathfrak{A} \iff q \in \delta'^*(\{q_0\}, w) \text{ in } \mathfrak{A}'$$

and thus

$$\mathfrak{A} \text{ accepts } w \iff \mathfrak{A}' \text{ accepts } w$$

□

From NFA to DFA II

Proof (continued).

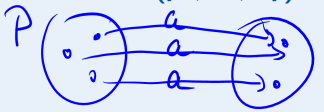
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$q_0' := \{q_0\}$

$F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$



$\delta'(P,a) = \{q \in Q \mid \exists p \in P \cdot p \xrightarrow{a} q\}$

This yields

$$q_0 \xrightarrow{w} q \text{ in } \mathfrak{A} \iff q \in \delta'^*(\{q_0\}, w) \text{ in } \mathfrak{A}'$$

and thus

$$\mathfrak{A} \text{ accepts } w \iff \mathfrak{A}' \text{ accepts } w$$

□

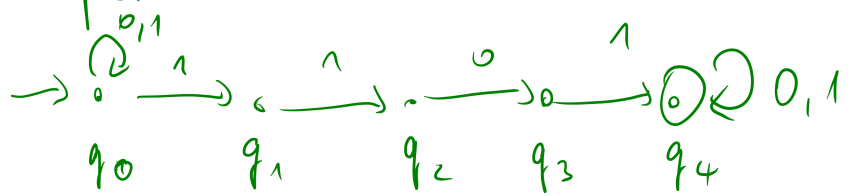
Example A.33

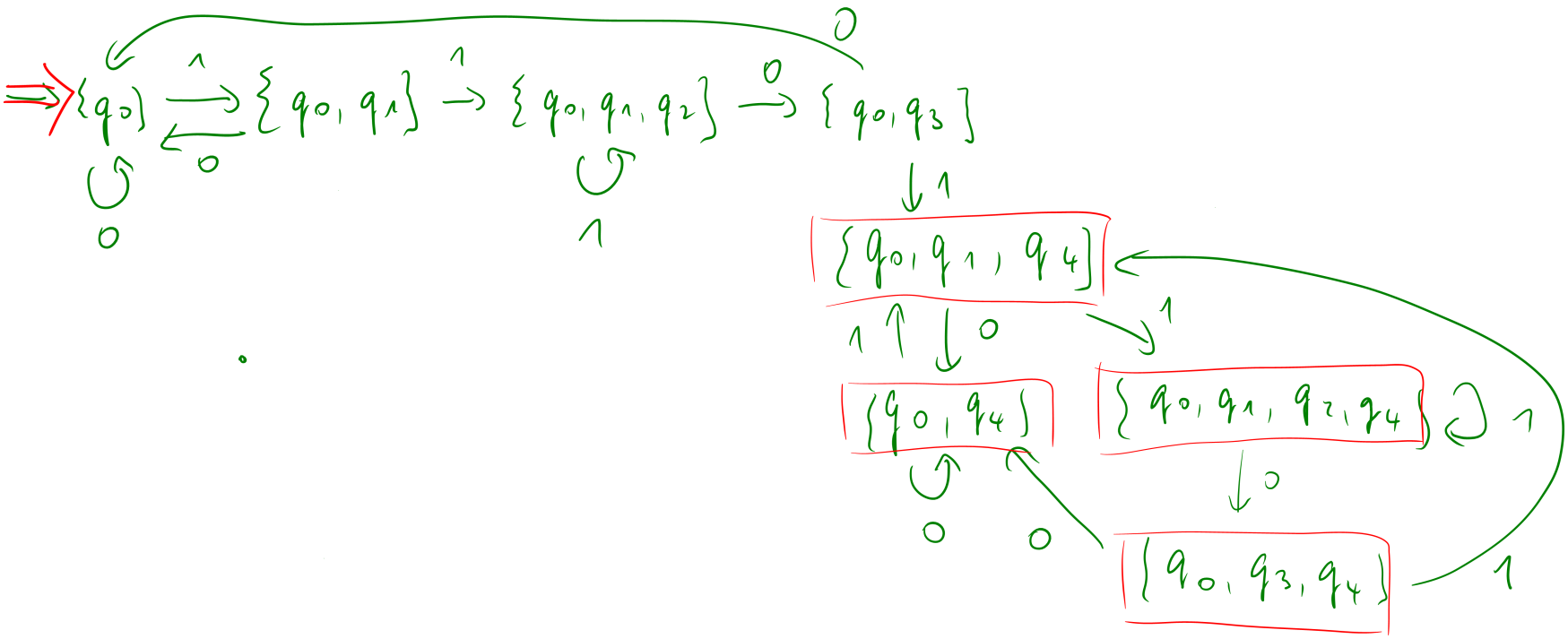
on the board

Example NFA → DFA

$\Sigma = \{0, 1\}$

$(0|1)^* 1 1 0 1 (0|1)^*$





From ε-NFA to NFA

Theorem A.34

Every ε-NFA can be transformed into an equivalent NFA.

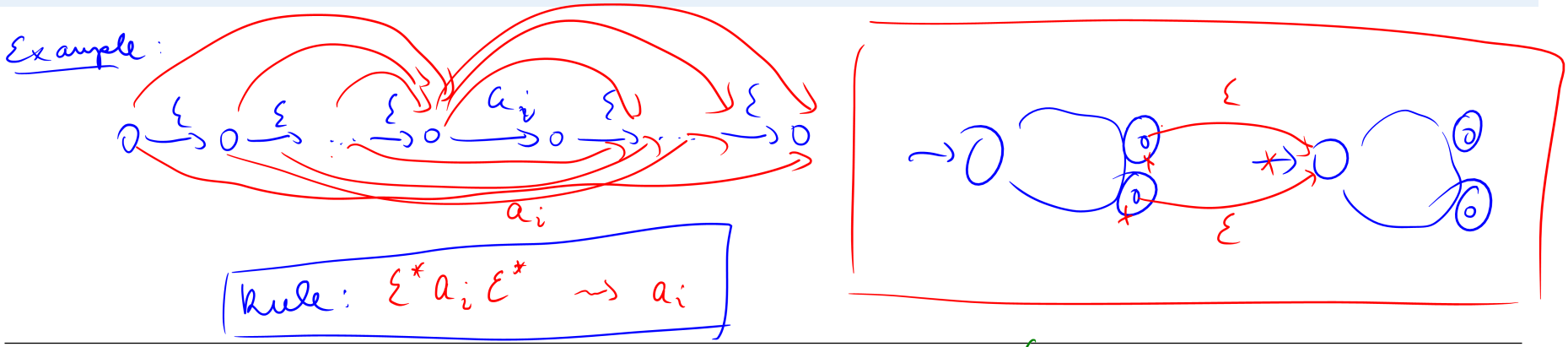
From ε-NFA to NFA

Theorem A.34

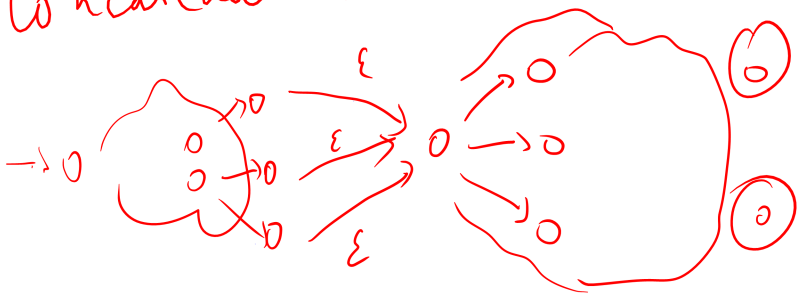
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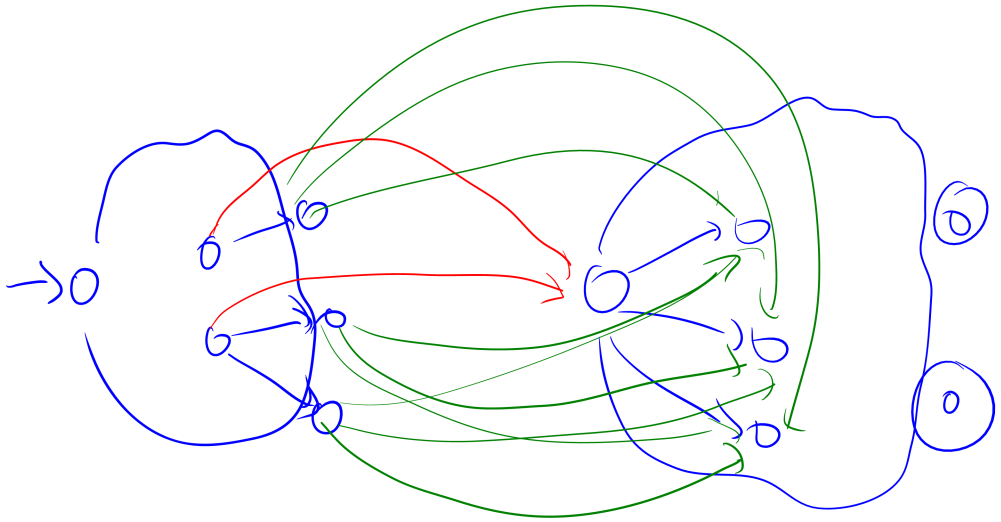
Proof (idea).

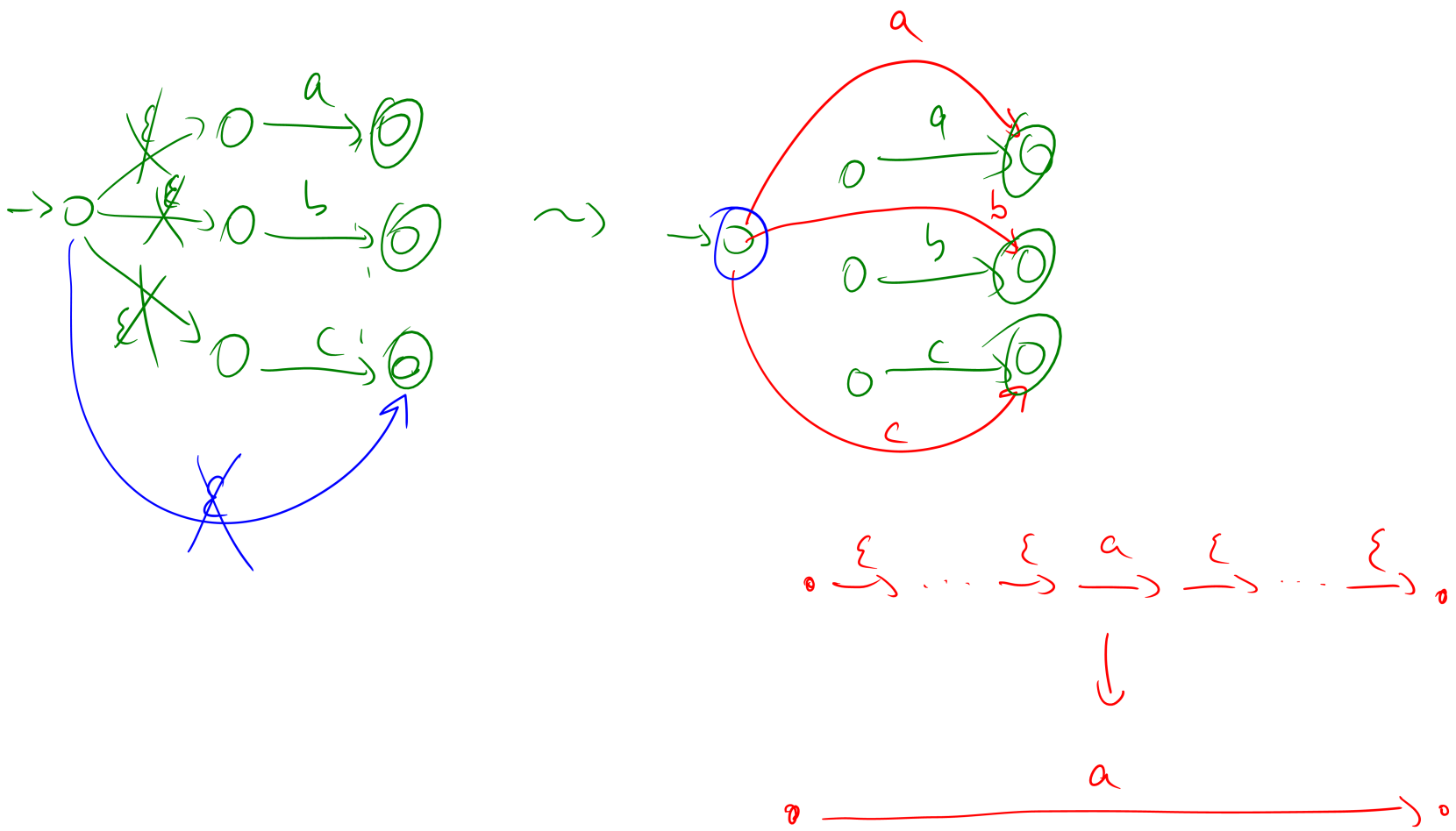
Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε-NFA. We construct the NFA $\mathfrak{A}'$ by eliminating all ε-transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}{}^{*} q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}{}^{*} r$ in $\mathfrak{A}$, then $p \xrightarrow{a} r$ in $\mathfrak{A}'$. Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}{}^{*} q \in F$ in $\mathfrak{A}$, and $F' := F$ otherwise. □



Concatenation






a
b
c
ε
ε
ε
a
b
c
a
b
c
ε
ε
ε
ε
a
ε
ε
a

From ε-NFA to NFA

Theorem A.34

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Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε-NFA. We construct the NFA $\mathfrak{A}'$ by eliminating all ε-transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}{}^{*} q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}{}^{*} r$ in $\mathfrak{A}$, then $p \xrightarrow{a} r$ in $\mathfrak{A}'$. Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}{}^{*} q \in F$ in $\mathfrak{A}$, and $F' := F$ otherwise. □

Example A.35

on the board

From ε-NFA to NFA

Theorem A.34

Every ε-NFA can be transformed into an equivalent NFA.

Proof (idea).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε-NFA. We construct the NFA $\mathfrak{A}'$ by eliminating all ε-transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}{}^{*} q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}{}^{*} r$ in $\mathfrak{A}$, then $p \xrightarrow{a} r$ in $\mathfrak{A}'$. Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}{}^{*} q \in F$ in $\mathfrak{A}$, and $F' := F$ otherwise. □

Example A.35

on the board

Corollary A.36

All three types of finite automata recognise the same class of languages.

Nondeterministic Finite Automata

Seen:

- Definition of (ε-)NFA
- Determinisation of (ε-)NFA

Nondeterministic Finite Automata

Seen:

Definition of (ε-)NFA

Determinisation of (ε-)NFA

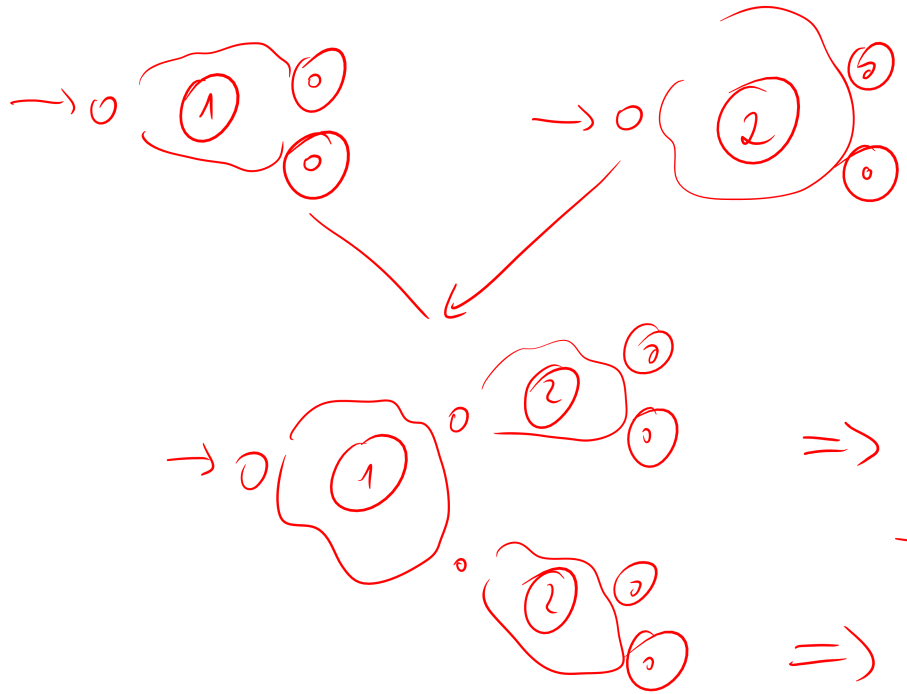
Open:

More decidablity results

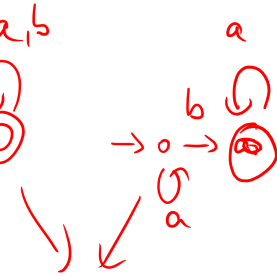
why did we introduce NFA and ε-NFA if they do not allow to model more?

Concatenation $\mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}(\mathcal{A}_2)$

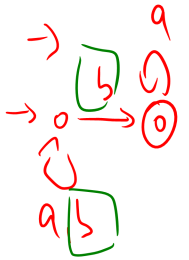
$$L_1 \cdot L_2 = \{ vw \mid v \in L_1 \wedge w \in L_2 \}$$



$\Rightarrow$ non-determinism



$\Rightarrow$ need ε-trans.

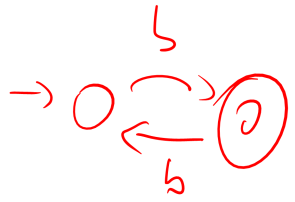




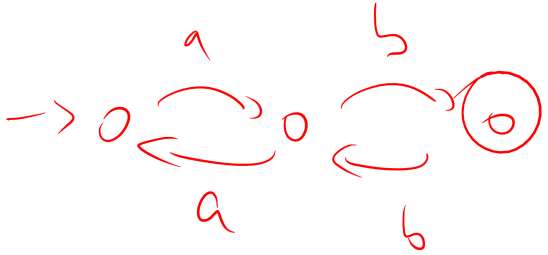
$\Rightarrow$ convenience of modelling





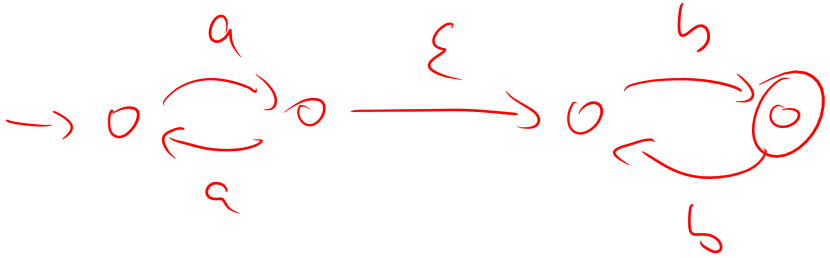


L_1 = odd number of a L_2 odd number of b $\Rightarrow L_1 \cdot L_2 = a(aa)^* b(bb)^*$



$\Rightarrow L = a\left((aa)^* \mid (bb)^*\right) b$

$a\,bb\,aa\,b$



Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

The Word Problem Revisited

Definition A.37

The **word problem for DFA** is specified as follows:

Given a DFA $\mathfrak{A}$ and a word $w \in \Sigma^*$, decide whether

$$w \in L(\mathfrak{A}).$$

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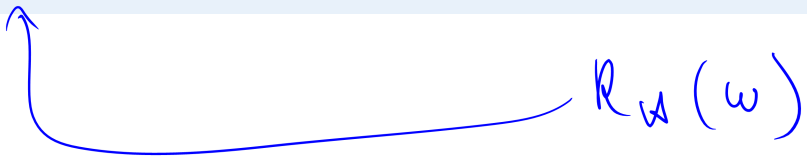
Given a DFA $\mathfrak{A}$ and a word $w \in \Sigma^*$, decide whether

$$w \in L(\mathfrak{A}).$$

As we have seen (Def. A.10, Alg. A.24, Thm. A.34):

Theorem A.38

*The word problem for DFA (NFA, ε-NFA) is **decidable**.*



The Emptiness Problem

Definition A.39

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Given a DFA $\mathfrak{A}$, decide whether $L(\mathfrak{A}) = \emptyset$.

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Proof.

It holds that $L(\mathfrak{A}) \neq \emptyset$ iff in $\mathfrak{A}$ some final state is reachable from the initial state (simple graph-theoretic problem). □

{} = ∅ – empty language

ε – empty word

{ε} ≠ ∅

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Remark: important result for formal verification (unreachability of bad [= final] states)

The Equivalence Problem

Definition A.41

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

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Theorem A.42

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Proof.

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Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff \quad & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \end{aligned}$$

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\iff\ & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) \cup (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset
\end{aligned}
$$

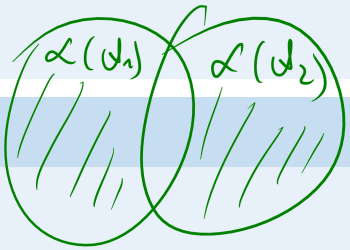
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\iff\ & (L(\mathfrak{A}_1) \cap \underbrace{\overline{L(\mathfrak{A}_2)}}_{\text{DFA-recognisable (Thm. A.14)}}) \cup (L(\mathfrak{A}_2) \cap \underbrace{\overline{L(\mathfrak{A}_1)}}_{\text{DFA-recognisable (Thm. A.14)}}) = \emptyset
\end{aligned}
$$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{DFA-recognisable (Thm. A.16)}} \quad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{DFA-recognisable (Thm. A.16)}}$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{DFA-recognisable (Thm. A.18)}}$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{decidable (Thm. A.40)}}$

□

Finite Automata

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Finite Automata

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Open:

- Non-algorithmic description of languages

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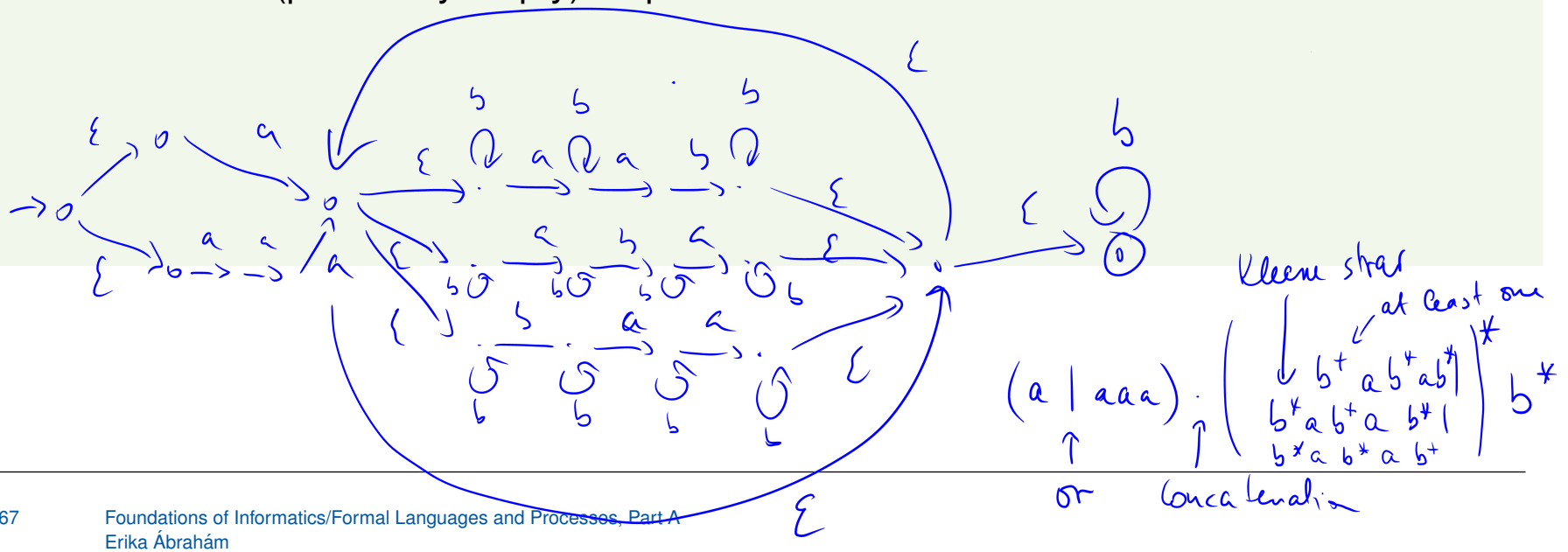
Outlook

An Example

Example A.43

Consider the set of all words over $\Sigma := \{a, b\}$ which

1. start with one or three a symbols
2. continue with a (potentially empty) sequence of blocks, each containing at least one b and exactly two a's
3. conclude with a (potentially empty) sequence of b's



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3. conclude with a (potentially empty) sequence of b's

Corresponding **regular expression**:

$$(a + aaa)(\underbrace{bb^*ab^*ab^*}_{b \text{ before } a\text{'s}} + \underbrace{b^*abb^*ab^*}_{b \text{ between } a\text{'s}} + \underbrace{b^*ab^*abb^*}_{b \text{ after } a\text{'s}})^*b^*$$

b^+

Syntax of Regular Expressions

Definition A.44

The set of **regular expressions** over Σ is inductively defined by:

$\emptyset$ and ε are regular expressions

every $a \in \Sigma$ is a regular expression

if α and β are regular expressions, then so are

- $\alpha + \beta$ $(\alpha + \beta)$ OR
- $\alpha \cdot \beta$ $(\alpha \cdot \beta)$ CONCATENATION
- α^* (α^*) KLEENE STAR (any repetition ≥ 0)

Syntax of Regular Expressions

Definition A.44

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- $\emptyset$ and ε are regular expressions
- every $a \in \Sigma$ is a regular expression
- if α and β are regular expressions, then so are
  - $\alpha + \beta$
  - $\alpha \cdot \beta$
  - α^*

Notation:

- $\cdot$ can be omitted
- * binds stronger than $\cdot$, $\cdot$ binds stronger than $+$
- α^+ abbreviates $\alpha \cdot \alpha^*$

$ab \rightsquigarrow a \cdot b$ $\quad ab^* \rightsquigarrow a(b^*)$ $\quad ab + c \rightsquigarrow (ab) + c$

α^+: at least one $\quad \alpha^k$: k times

Semantics of Regular Expressions

Definition A.45

Every regular expression α defines a language $L(\alpha)$:

$$\begin{aligned} L(\emptyset) &:= \emptyset \\ L(\varepsilon) &:= \{\varepsilon\} \\ L(a) &:= \{a\} \\ L(\alpha + \beta) &:= L(\alpha) \cup L(\beta) \\ L(\alpha \cdot \beta) &:= L(\alpha) \cdot L(\beta) \\ L(\alpha^*) &:= (L(\alpha))^* \end{aligned}$$

Semantics of Regular Expressions

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A language L is called **regular** if it is definable by a regular expression, i.e., if $L = L(\alpha)$ for some regular expression α.

Regular Languages

Example A.46

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

Regular Languages

Example A.46

1. $\{aa\}$ is regular since

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2. $\{a, b\}^*$ is regular since

$$L((a+b)^*) = (L(a+b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

$$= \{ \varepsilon, a, b, ab, ba, aab, \ldots \}$$

Regular Languages

Example A.46

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

2. $\{a, b\}^*$ is regular since

$$L((a+b)^*) = (L(a+b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

3. The set of all words over $\{a, b\}$ containing abb is regular since

$$L((a+b)^* \cdot a \cdot b \cdot b \cdot (a+b)^*) = \{a, b\}^* \cdot \{abb\} \cdot \{a, b\}^*$$

Regular Languages and Finite Automata I

Theorem A.47 (Kleene's Theorem)

To each regular expression there corresponds an ε-NFA, and vice versa.

Regular Languages and Finite Automata I

Theorem A.47 (Kleene's Theorem)

To each regular expression there corresponds an ε-NFA, and vice versa.

Proof.

$\Rightarrow$: using induction over the given regular expression α, we construct an ε-NFA $\mathfrak{A}_\alpha$

- with exactly one final state q_f
- without transitions into the initial state
- without transitions leaving the final state

(on the board)

$\Leftarrow$: by solving a regular equation system (details omitted)

□

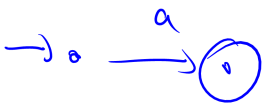
ϕ



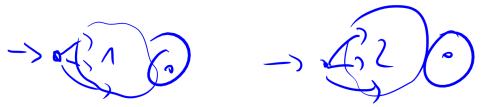
ε

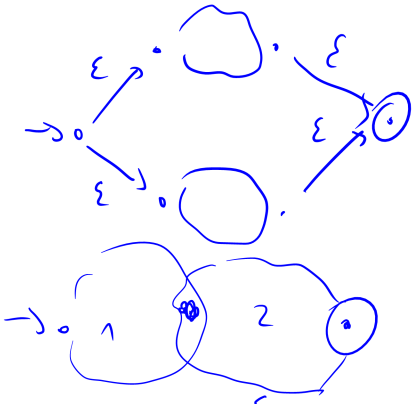


a



$e_1 + e_2$

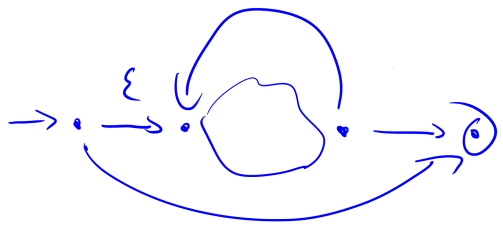


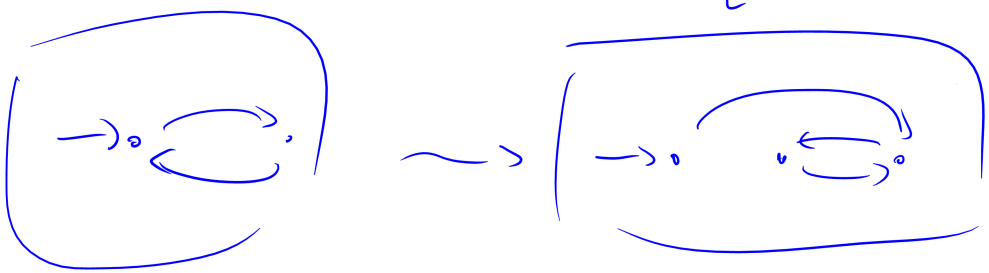


$e_1 \cdot e_2$

e^*







Example: $(a+b)^* \cdot b \cdot (a+b)$

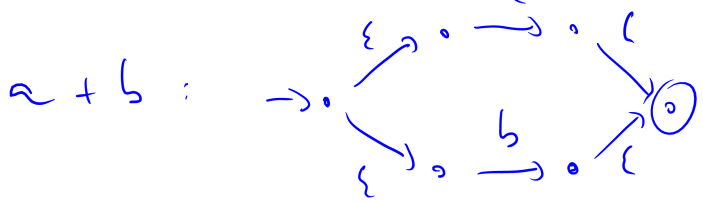
a:



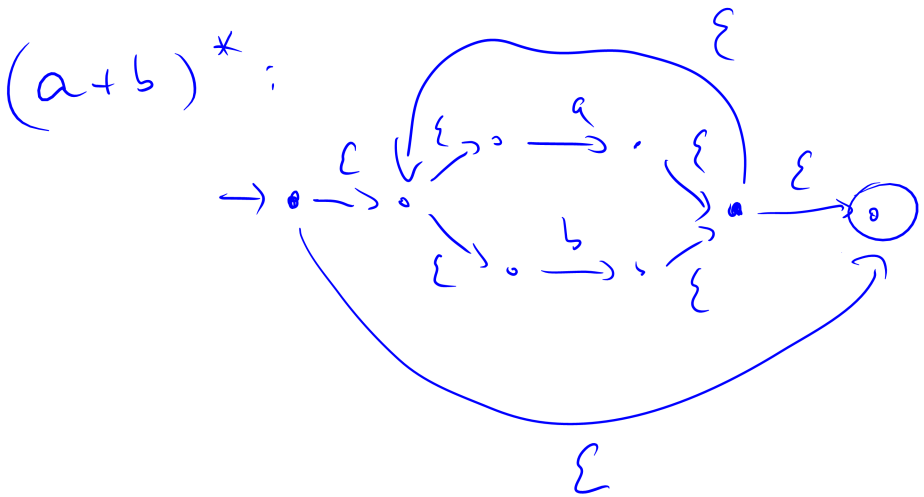
b:



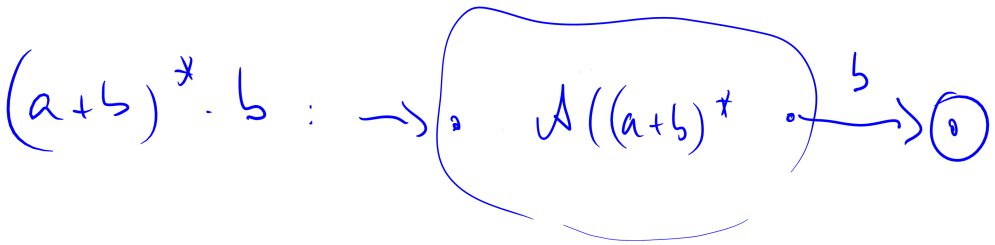
$a+b$:



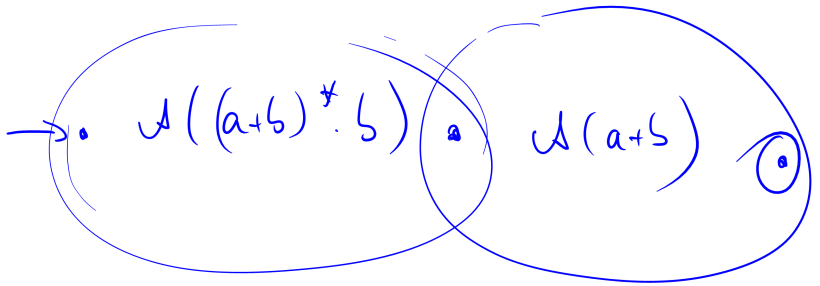
$(a+b)^*$:

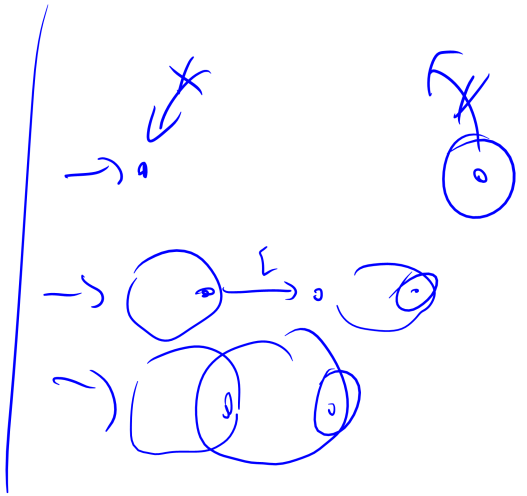


$(a+b)^* \cdot b$:

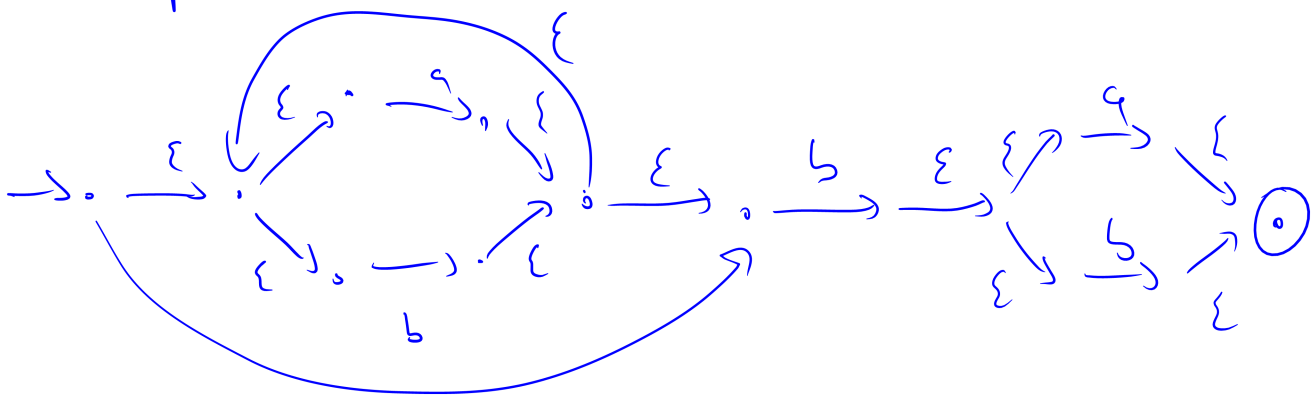


$(a+b)^* \cdot b \cdot (a+b)$:



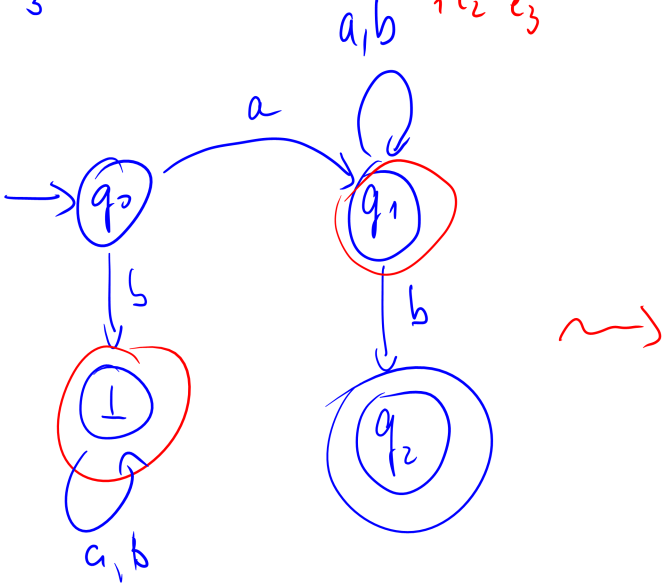


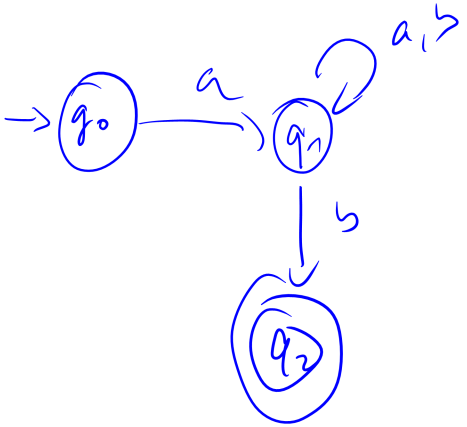
The final automaton: ε-NFA

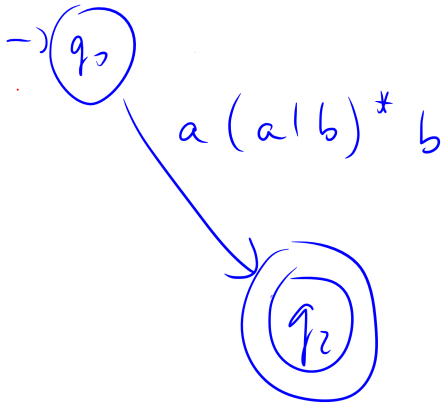




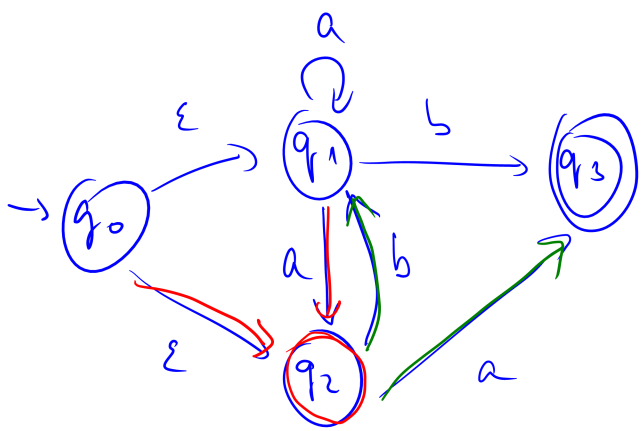
Example for ε-NFA to regular expr.



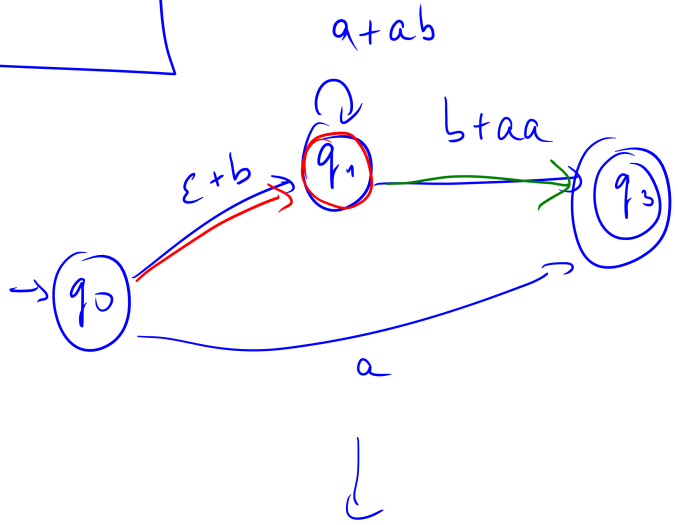




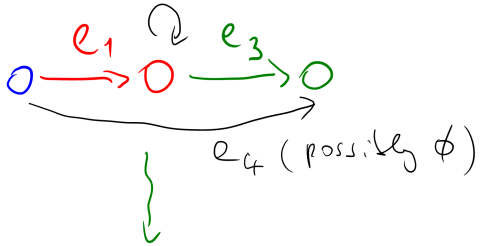
Example 2 for ε-NFA $\rightarrow$ reg. expr.

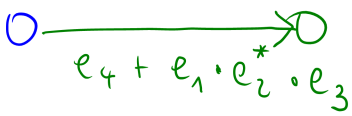


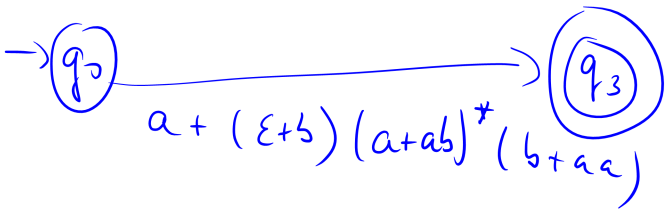
$\rightsquigarrow$



Rule : $e_2 \leftarrow$ possibly ϕ







Regular Languages and Finite Automata II

Corollary A.48

The following properties are equivalent:

- *L is regular*
- *L is DFA-recognisable*
- *L is NFA-recognisable*
- *L is ε-NFA-recognisable*

Implementation of Pattern Matching

Algorithm A.49 (Pattern Matching)

Input: *regular expression* α *and* $w \in \Sigma^*$

Question: *does* w *contain some* $v \in L(\alpha)$*?*

Procedure:
1. *let* $\beta := (a_1 + \ldots + a_n)^* \cdot \alpha$ *(for* $\Sigma = \{a_1, \ldots, a_n\}$*)*
2. *determine* ε*-NFA* $\mathfrak{A}_\beta$ *for* β
3. *eliminate* ε*-transitions*
4. *apply powerset construction to obtain DFA* $\mathfrak{A}$
5. *let* $\mathfrak{A}$ *run on* w

Output: *"yes" if* $\mathfrak{A}$ *passes through some final state, otherwise "no"*

Remark: in UNIX/LINUX implemented by `grep` and `lex`

Regular Expressions in UNIX (`grep`, `flex`, ...)

Syntax	Meaning
printable character	this character
`\n`, `\t`, `\123`, etc.	newline, tab, octal representation, etc.
`.`	any character except `\n`
[*Chars*]	one of *Chars*; ranges possible ("`0-9`")
[`^`*Chars*]	none of *Chars*
`\\`, `\.`, `\[`, etc.	`\`, `.`, `[`, etc.
"*Text*"	*Text* without interpretation of `.`, `[`, `\`, etc.
`^`α	α at beginning of line
α`$`	α at end of line
α`?`	zero or one α
α`*`	zero or more α
α`+`	one or more α
α{*n*, *m*}	between *n* and *m* times α (", *m*" optional)
(α)	α
$\alpha_1\alpha_2$	concatenation
α_1`|`α_2	alternative

Regular Expressions

Seen:

Definition of regular expressions

Equivalence of regular and DFA-recognisable languages

Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

Minimisation of DFA

Motivation

Goal: space-efficient implementation of regular languages

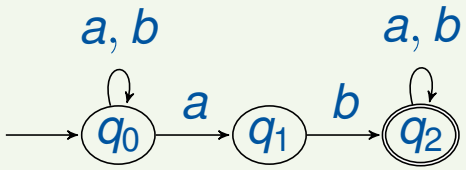
Given: DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$

Wanted: DFA $\mathfrak{A}_{min} = \langle Q', \Sigma, \delta', q_0', F' \rangle$ such that $L(\mathfrak{A}_{min}) = L(\mathfrak{A})$ and $|Q'|$ **minimal**

State Equivalence

Example A.50

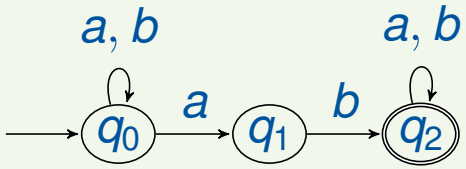
NFA for accepting $(a+b)^*ab(a+b)^*$:



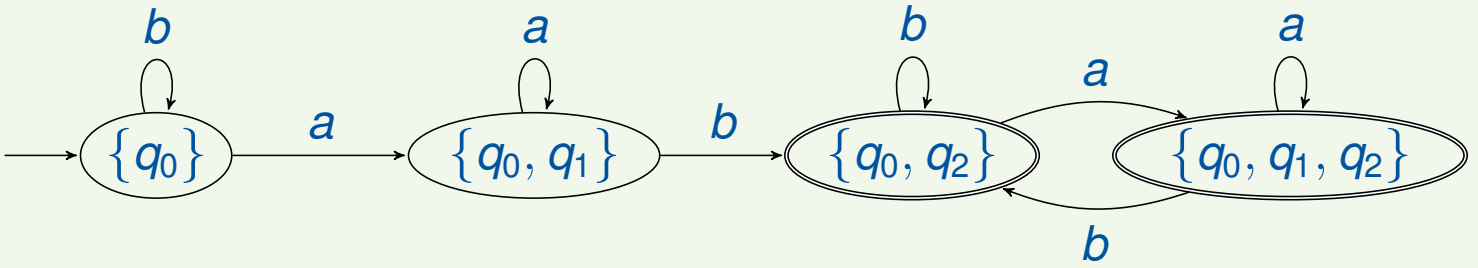
State Equivalence

Example A.50

NFA for accepting $(a+b)^*ab(a+b)^*$:



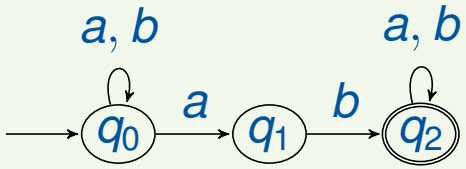
Powerset construction yields DFA $\mathfrak{A}$:



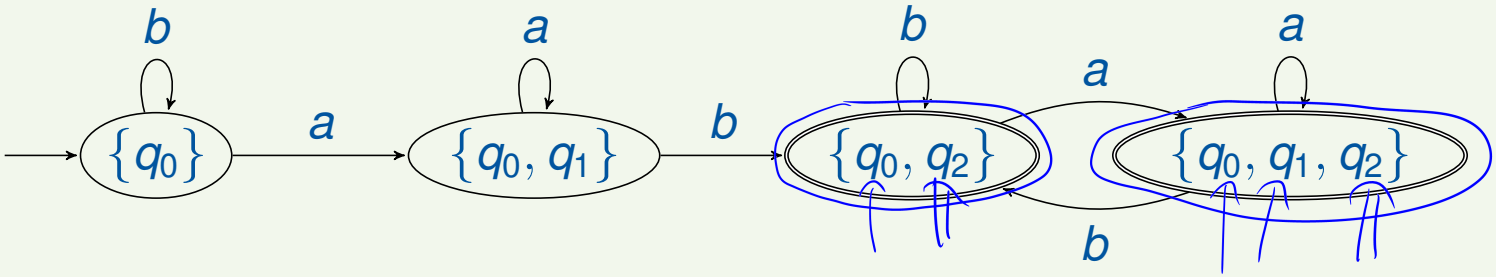
State Equivalence

Example A.50

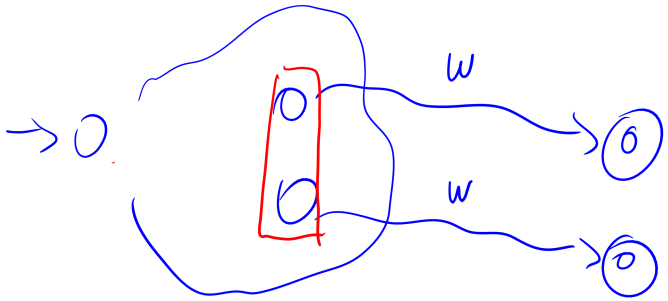
NFA for accepting $(a+b)^*ab(a+b)^*$:



Powerset construction yields DFA $\mathfrak{A}$:



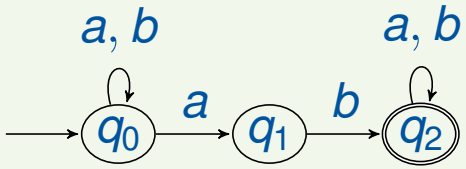
Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are **equivalent**


w
w

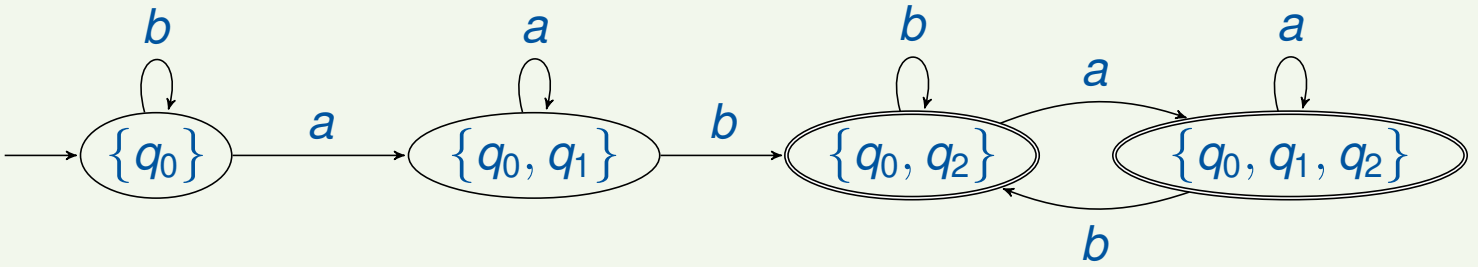
State Equivalence

Example A.50

NFA for accepting $(a+b)^*ab(a+b)^*$:



Powerset construction yields DFA $\mathfrak{A}$:



Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are **equivalent**

Definition A.51

Given DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$, states $p, q \in Q$ are **equivalent** if

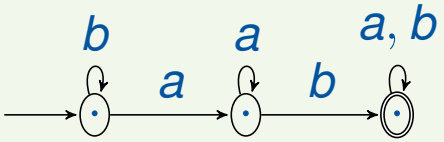
$$\forall w \in \Sigma^* : \delta^*(p, w) \in F \iff \delta^*(q, w) \in F.$$

Minimisation

Minimisation: **merging** of equivalent states

Example A.52 (cf. Example A.50)

DFA after state merging:

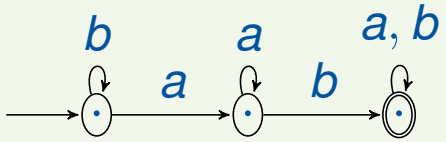


Minimisation

Minimisation: **merging** of equivalent states

Example A.52 (cf. Example A.50)

DFA after state merging:



Problem: **identification** of equivalent states

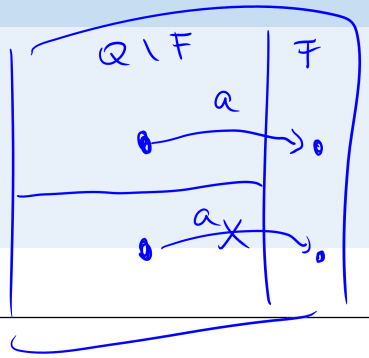
Approach: iterative computation of inequivalent states by refinement

Corollary A.53

*$p, q \in Q$ are **inequivalent** if there exists $w \in \Sigma^*$ such that*

$$\delta^*(p, w) \in F \text{ and } \delta^*(q, w) \notin F$$

(or vice versa, i.e., p and q can be distinguished by w)



Computing State (In-)Equivalence

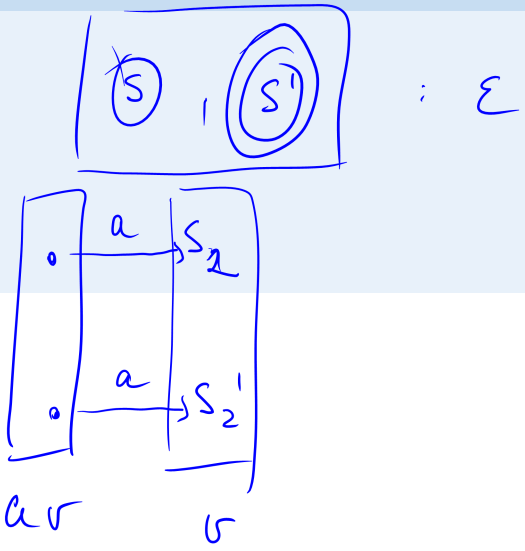
Lemma A.54

Inductive characterisation of state inequivalence:

$w = \varepsilon$: $p \in F, q \notin F \implies p, q$ *inequivalent (by* ε*)*

$w = av$: p', q' *inequivalent (by* v*),* $p \xrightarrow{a} p', q \xrightarrow{a} q'$

$\implies p, q$ *inequivalent (by* w*)*



Computing State (In-)Equivalence

Lemma A.54

Inductive characterisation of state inequivalence:

$w = \varepsilon$: $p \in F, q \notin F \implies p, q$ *inequivalent (by* ε*)*

$w = av$: p', q' *inequivalent (by* v*),* $p \xrightarrow{a} p', q \xrightarrow{a} q'$

$\implies p, q$ *inequivalent (by* w*)*

Algorithm A.55 (State Equivalence for DFA)

Input: *DFA* $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$

Procedure: *Computation of* ***"equivalence matrix"*** *over* $Q \times Q$

1. *mark every pair* (p, q) *with* $p \in F, q \notin F$ *by* ε
2. *for every unmarked pair* (p, q) *and every* $a \in \Sigma$*:*
   if $(\delta(p, a), \delta(q, a))$ *marked by* v*, then mark* (p, q) *by* av
3. *repeat until no change*

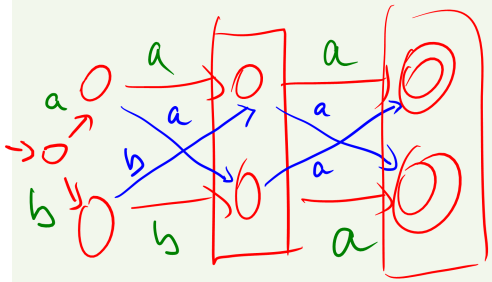
Output: *all equivalent (= unmarked) pairs of states*

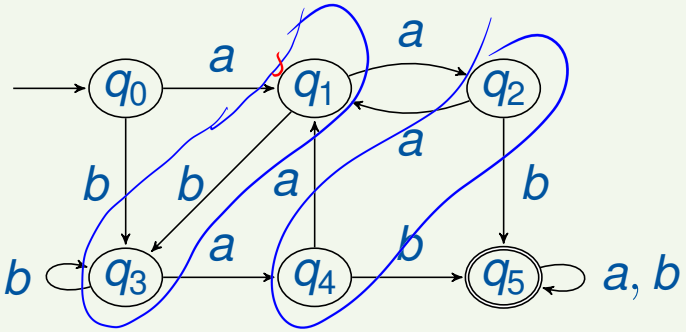
For each $(q_1, q_2) \in Q \times Q$ and each $a \in \Sigma$ with no entry yet check whether $(\delta(q_1, a), \delta(q_2, a))$ has entry: no → –; yes → add entry a, v for v

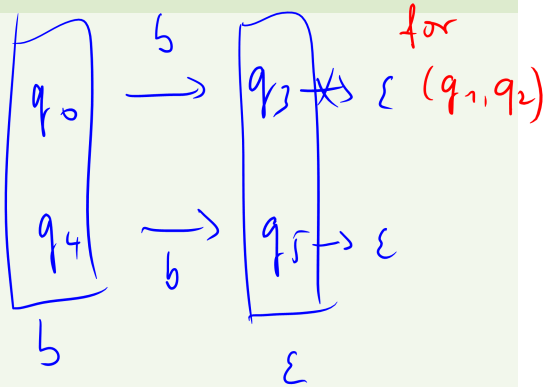
Minimisation Example

Example A.56

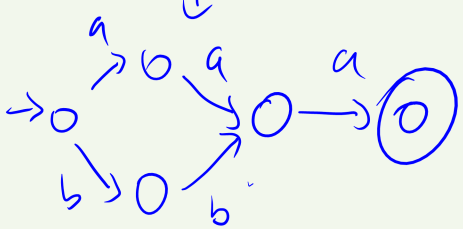
Given DFA:

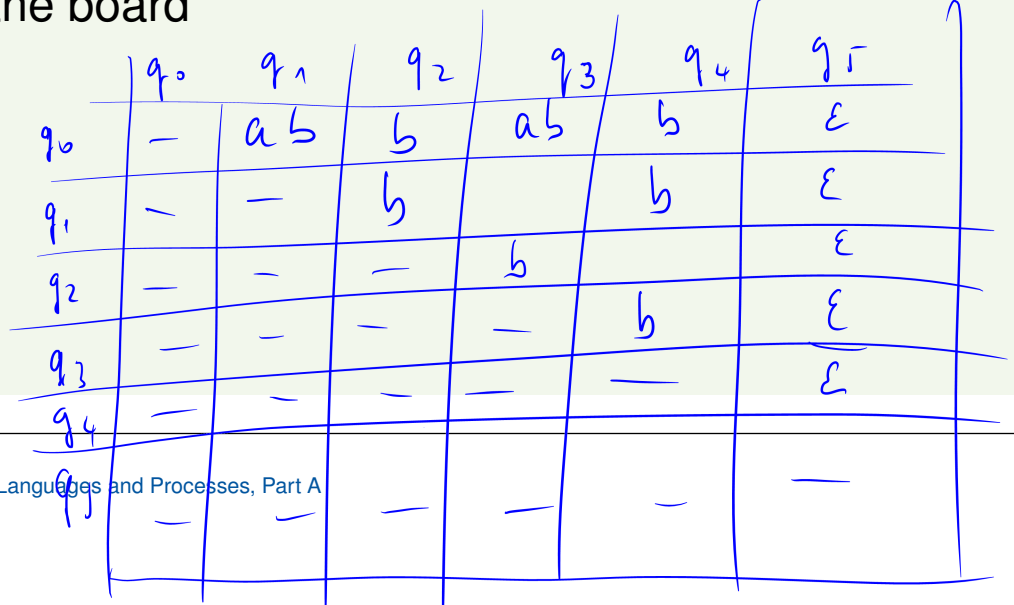


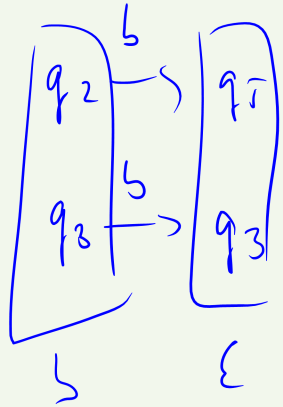




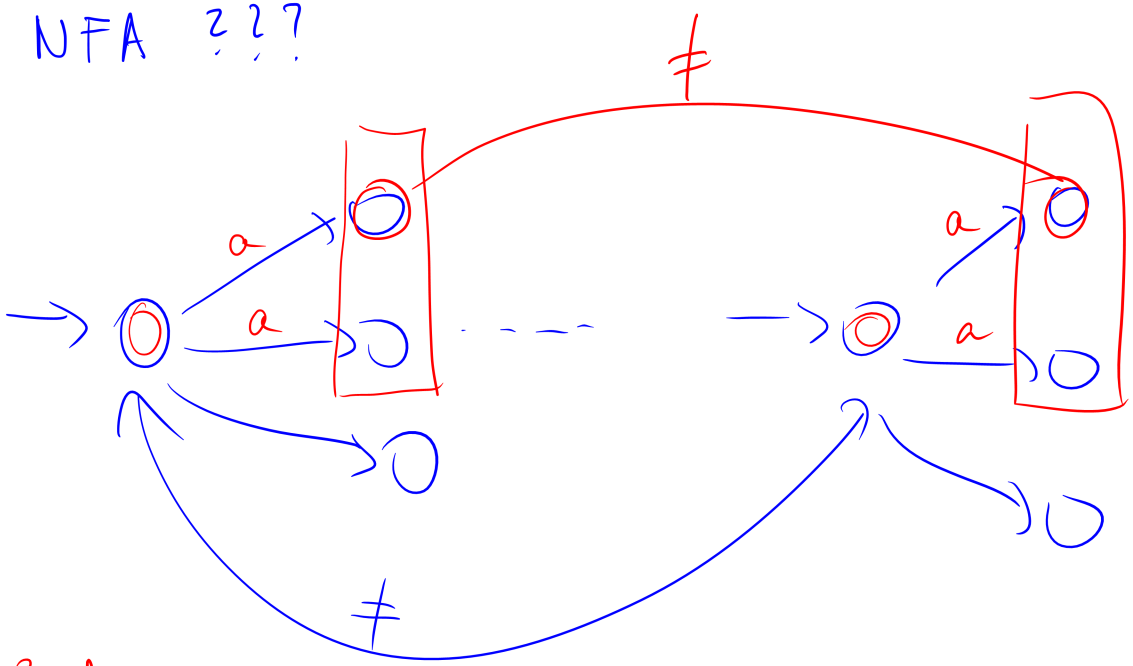
Equivalence matrix: on the board







Applicable also to minimize NFA ???



Probably extensible to NFA (and ε-NFA)

But

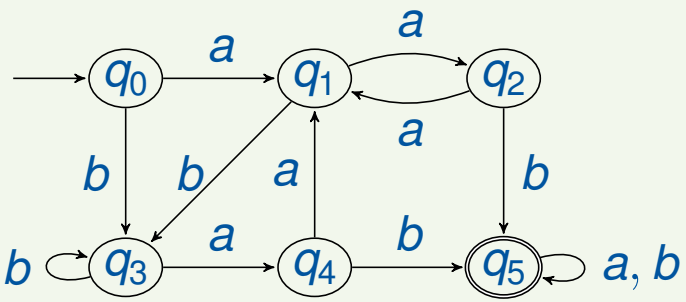
Safe solution

ε-NFA → NFA → DFA → minimize

Minimisation Example

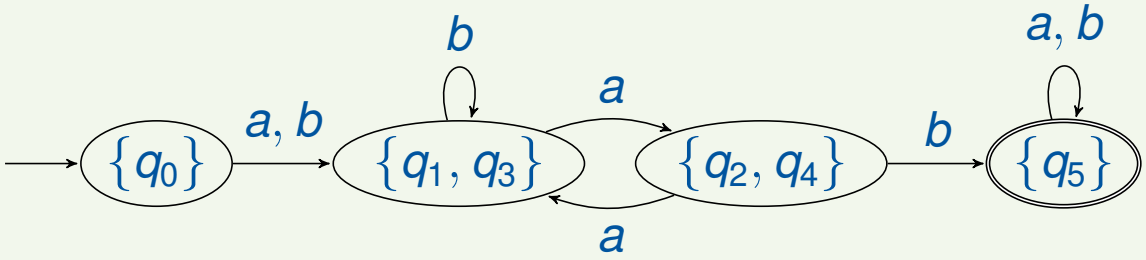
Example A.56

Given DFA:



Equivalence matrix: on the board

Resulting minimal DFA:



Correctness of Minimisation

Theorem A.57

For every DFA $\mathfrak{A}$,

$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Correctness of Minimisation

Theorem A.57

For every DFA $\mathfrak{A}$,

$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Remark: the minimal DFA is **unique**, in the following sense:

$$\forall \text{DFA } \mathfrak{A}, \mathfrak{B} : L(\mathfrak{A}) = L(\mathfrak{B}) \implies \mathfrak{A}_{min} \approx \mathfrak{B}_{min}$$

where $\approx$ refers to automata isomorphism (= identity up to naming of states)

Outline of Part A

Formal Languages

Finite Automata
- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

Minimisation of DFA

Outlook

Outlook

Pumping Lemma (to prove non-regularity of languages)

– can be used to show that $\{a^n b^n \mid n \geq 1\}$ is not regular

More **language operations** (homomorphisms, ...)

Construction of **scanners** for compilers